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Diagonal Lyapunov–Krasovskii functionals for discrete-time positive systems with delay

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HIGHLIGHTS

- Conditions for diagonal Lyapunov–Krasovskii functional existence are given.
- The connection between diagonal L–K functionals and absolute stability is shown.
- The previous facts are extended to nonlinear and switched systems.

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ABSTRACT

We consider the existence of diagonal Lyapunov–Krasovskii (L–K) functionals for positive discrete-time systems subject to time-delay. In particular, we show that the existence of a diagonal functional is necessary and sufficient for delay-independent stability of a positive linear time-delay system. We extend this result and provide conditions for the existence of diagonal L–K functionals for classes of nonlinear positive time-delay systems, which are not necessarily order preserving. We also describe sufficient conditions for the existence of common diagonal L–K functionals for switched positive systems subject to time-delay.

1. Introduction

Positive systems, in which the state variables remain nonnegative for all time given nonnegative initial conditions, are of importance in modeling a wide variety of applications; particularly in domains such as Communications, Biology, Ecology and Economics. There is now a well-developed and understood theory of positive linear time-invariant (LTI) systems [1,2]. Motivated by the simple fact that realistic models must incorporate factors such as nonlinearity, time-delay and time-varying parameters, several authors have recently worked on extending aspects of the theory of positive LTI systems to more realistic and general system classes [3–8]. The work of the current paper continues in this vein.

Specifically, we consider classes of discrete time positive systems and the impact of time-delay on their stability properties. We first note that a recently published result on the existence of diagonal L–K functionals for positive linear systems in continuous time admits a natural analogue in discrete time. In fact, we explicitly describe how to construct such functionals in the discrete time case. This result, as in [9], shows that a fundamental property of positive LTI systems, namely diagonal stability, extends to positive linear time-delay systems in discrete time.

In Section 4, we use the results in Section 3 to derive a closely related condition for positive systems with sector-bounded nonlinearities to be stable independent of delay. It is worth noting that this result applies to nonlinear positive systems that are not necessarily order preserving, in contrast to others in the literature. It is closely related to various results for systems of Persidskii type described in [10] and elsewhere. As noted in this reference, these systems arise in applications such as digital filtering. We then turn our attention to nonlinear switched positive systems with time-delay in Section 5. Finally, in Section 6, we present some brief concluding remarks.

2. Notation and background

Throughout the paper $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ denote the vector spaces of n-tuples of real numbers and of $n\times n$ matrices respectively. For vectors $v \in \mathbb{R}^n$, $v \geq 0$ means $v_i \geq 0$ for $1 \leq i \leq n$, $v \gg 0$ means $v_i > 0$ for $1 \leq i \leq n$. We use the notation A^T for the transpose of a matrix A and $P \succ 0$ to denote that the matrix P is positive definite. The

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identity matrix is denoted by I; the dimension will be clear in context. We say that a matrix $A \in \mathbb{R}^{n\times n}$ is nonnegative if all of its elements are nonnegative. The matrix A is *Schur* if all of its eigenvalues have modulus strictly less than 1.

The following known facts about nonnegative matrices are useful for our later results.

Proposition 2.1. *Let $A \in \mathbb{R}^{n\times n}$ be nonnegative. The following are true.*

(i) *A is Schur if and only if there exists some $v \gg 0$ with $Av \ll v$.*
(ii) *If $Av \ll v$ and $A^T w \ll w$ then defining*

$$D = \mathrm{diag}(w_i/v_i)$$

we have

$$A^T DA - D \prec 0.$$

3. Diagonal L–K functionals for linear discrete time delayed systems

Consider the positive linear time-delay system

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) \tag{1}$$

where A is Metzler and B is nonnegative. It is now well known that (1) is asymptotically stable for all values of $\tau \geq 0$ if and only if $A+B$ is Hurwitz [11], meaning that all of its eigenvalues lie in the open left half of the complex plane.

In the recent paper [9], the following fact was established.

Theorem 3.1. *The positive linear time-delay system (1) has a L–K functional of the form*

$$x(t)^T Px(t) + \int_{-\tau}^{0} x(t+s)Qx(t+s)ds$$

where P and Q are positive definite matrices and P is diagonal, if and only if the matrix $A+B$ is Hurwitz stable.

In the current section, we derive a discrete time version of Theorem 3.1.

Formally, consider the system

$$x(k+1) = Ax(k) + B_1x(k-1) + \cdots + B_lx(k-l), \tag{2}$$

where $A, B_1, \ldots, B_l$ are nonnegative matrices in $\mathbb{R}^{n\times n}$. Under these assumptions (2) defines a positive time-delay system in discrete time, meaning that if initial conditions $x(-l), \ldots, x(0)$ are nonnegative then $x(k) \geq 0$ for all $k \geq 0$. For notational convenience, we write $x^{(k)}$ for the augmented state vector $x^{(k)} = (x(k), x(k-1), \ldots, x(k-l))^T$ in $\mathbb{R}^{(l+1)n}$.

In the following result, we present a direct argument to establish a discrete-time version of Theorem 3.1.

Theorem 3.2. *Consider the system (2) and assume that the matrix S given by*

$$S = A + B_1 + \cdots + B_l \tag{3}$$

is Schur. Then there exists a L–K functional for (2) of the form

$$\begin{aligned} V(x^{(k)}) = {} & x^T(k)Px(k) + x^T(k-1)Q_1x(k-1) \\ & + \{x^T(k-1)Q_2x(k-1) + x^T(k-2)Q_2x(k-2)\} \\ & + \cdots + \{x^T(k-1)Q_lx(k-1) + x^T(k-2) \\ & \times Q_lx(k-2) + \cdots + x^T(k-l)Q_lx(k-l)\} \end{aligned} \tag{4}$$

where $P, Q_1, \ldots, Q_l$ are positive definite diagonal matrices.

Proof. We write $P = \mathrm{diag}\{p_1, \ldots, p_n\}$, $Q_s = \mathrm{diag}\{\mu_{s1}, \ldots, \mu_{sn}\}$ for the matrices appearing in (4). We shall show that it is possible to find positive real numbers p_i and μ_{si}, $s = 1, \ldots, l$, $i = 1, \ldots, n$ such that (4) is a L–K functional for the system (2).

Consider the difference, $\Delta V = V(x^{(k+1)}) - V(x^{(k)})$ of the functional (4) along trajectories of the system (2). We obtain

$$\begin{aligned} \Delta V = {} & x^T(k)\left(A^TPA - P + \sum_{r=1}^{l} Q_r\right)x(k) \\ & - \sum_{r=1}^{l} x^T(k-r)Q_rx(k-r) + 2x^T(k)A^TP\sum_{r=1}^{l} B_rx(k-r) \\ & + \left(\sum_{r=1}^{l} B_rx(k-r)\right)^T P\sum_{r=1}^{l} B_rx(k-r). \end{aligned}$$

Expanding this yields

$$\begin{aligned} \Delta V = {} & \sum_{r=1}^{l}\sum_{j=1}^{n} \mu_{rj}x_j^2(k) - \sum_{j=1}^{n} p_jx_j^2(k) \\ & - \sum_{r=1}^{l}\sum_{j=1}^{n} \mu_{rj}x_j^2(k-r) + \sum_{i,j=1}^{n} x_i(k)x_j(k)\sum_{m=1}^{n} p_m a_{mi}a_{mj} \\ & + 2\sum_{r=1}^{l}\sum_{i,j=1}^{n} x_i(k)x_j(k-r)\sum_{m=1}^{n} p_m a_{mi}b_{mj}^{(r)} \\ & + \sum_{r,s=1}^{l}\sum_{i,j=1}^{n} x_i(k-r)x_j(k-s)\sum_{m=1}^{n} p_m b_{mi}^{(r)}b_{mj}^{(s)}. \end{aligned}$$

The expression on the right hand side above defines a quadratic form

$$\begin{aligned} W = {} & \sum_{r=1}^{l}\sum_{j=1}^{n} \mu_{rj}x_j^2 - \sum_{j=1}^{n} p_jx_j^2 - \sum_{r=1}^{l}\sum_{j=1}^{n} \mu_{rj}x_{rj}^2 \\ & + \sum_{i,j=1}^{n} x_ix_j\sum_{m=1}^{n} p_m a_{mi}a_{mj} + 2\sum_{r=1}^{l}\sum_{i,j=1}^{n} x_ix_{rj} \\ & \times \sum_{m=1}^{n} p_m a_{mi}b_{mj}^{(r)} + \sum_{r,s=1}^{l}\sum_{i,j=1}^{n} x_{ri}x_{sj}\sum_{m=1}^{n} p_m b_{mi}^{(r)}b_{mj}^{(s)} \end{aligned}$$

in the variables x_i, $1 \leq i \leq n$, x_{ri}, $1 \leq i \leq n$, $1 \leq r \leq l$. Our goal is to show that there exist positive numbers p_i and μ_{si}, $s = 1, \ldots, l$, $i = 1, \ldots, n$, for which the quadratic form W is negative definite.

As the matrix S is Schur, it follows from Proposition 2.1 that there exist positive vectors $\theta = (\theta_1, \ldots, \theta_n)^T$ and $d = (d_1, \ldots, d_n)^T$ satisfying the inequalities

$$(S - I)\theta \ll 0, \qquad (S - I)^Td \ll 0. \tag{5}$$

Write $\omega = (S - I)\theta$, $\psi = (S - I)^Td$, and note that the vectors ω and ψ are entrywise negative.

If we write, $p_i = d_i/\theta_i$, $x_i = \theta_iz_i$, $x_{ri} = \theta_iv_{ri}$, then we can rewrite W as

$$\begin{aligned} W = {} & \sum_{r=1}^{l}\sum_{j=1}^{n} \mu_{rj}\theta_j^2z_j^2 - \sum_{j=1}^{n} d_j\theta_jz_j^2 - \sum_{r=1}^{l}\sum_{j=1}^{n} \mu_{rj}\theta_j^2v_{rj}^2 \\ & + \sum_{i,j=1}^{n} \theta_iz_i\theta_jz_j\sum_{m=1}^{n} \frac{d_m}{\theta_m}a_{mi}a_{mj} + 2\sum_{r=1}^{l}\sum_{i,j=1}^{n} \theta_iz_i\theta_jv_{rj} \\ & \times \sum_{m=1}^{n} \frac{d_m}{\theta_m}a_{mi}b_{mj}^{(r)} + \sum_{r,s=1}^{l}\sum_{i,j=1}^{n} \theta_iv_{ri}\theta_jv_{sj}\sum_{m=1}^{n} \frac{d_m}{\theta_m}b_{mi}^{(r)}b_{mj}^{(s)}. \end{aligned}$$