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Preservation of quadratic stability under various common approximate discretization methods

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ABSTRACT

In this paper we prove the following result. If A is a Hurwitz matrix and f is a rational function that maps the open left half of the complex plane into the open unit disc, then any Hermitian matrix $P > 0$ which is a Lyapunov matrix for A (that is, $PA + A^*P < 0$) is also a Stein matrix for $f(A)$ (that is, $f(A)^*Pf(A) - P < 0$).

We use this result to prove that all A-stable approximations for the matrix exponential preserve quadratic Lyapunov functions for any stable linear system. The importance of this result is that it implies that common quadratic Lyapunov functions for switched linear systems are preserved for all step sizes when discretising quadratically stable switched systems using A-stable approximations.

Examples are given to illustrate our results.

1. Introduction

Discretisation methods play an important role in the simulation and digital control of physical systems. Thus, it is of interest to explore how a continuous-time system can be transformed into a discrete-time system in a manner that preserves certain properties of the original system. In this paper, we focus on the problem of discretising a switched linear system of the form

$$\Sigma_{sc} : \dot{x} = A_c(t)x, \qquad A_c(t) \in \mathscr{A}_c, \tag{1}$$

where $\mathscr{A}_c$ is a finite set of matrices, into an approximate discrete-time counterpart

$$\Sigma_{sc} : x(k+1) = A_d(k)x(k), \quad A_d(k) \in \mathscr{A}_d, \tag{2}$$

where $\mathscr{A}_d$ is a finite set of matrices, subject to the constraint that exponential stability is preserved.

Discretisation of linear switched systems has been studied by a number of authors [1–8] and [9]. Roughly speaking, these and other authors have focused on developing discretisation methods that preserve certain system dependent constraints. Results in this direction include methods that not only give good approximations to the matrix exponential but also preserve properties such as positivity, sparseness as well as certain types of stability. Our work on this topic has focused on preserving common quadratic Lyapunov functions [4]. In this paper, we complete this initial work and prove that all A-stable discretisation methods preserve quadratic stability.

Our note is structured as follows. First, we give preliminary definitions and results. Then, we present our main result and discuss its implications. Finally, we provide numerical examples to illustrate the usefulness of the results.

2. Preliminaries

We now present some preliminary results and definitions that are useful in this note.

1. *Notation*: The real part of a complex number z is denoted by $\Re(z)$. The symbols $\mathbb{R}$ and $\mathbb{C}$ denote the set of real and complex numbers, respectively. The conjugate of a complex number z is denoted by z^* and M^* denotes the conjugate transpose of a general matrix M. A matrix P is Hermitian if $P^* = P$. A Hermitian matrix P is positive (negative) definite if $x^*Px > 0$ ($x^*Px < 0$) for all non-zero x and we denote this by $P > 0$ ($P < 0$).

2. *Hurwitz and Schur stability*: consider a continuous-time LTI (linear time-invariant) system given by

$$\Sigma_c : \dot{x} = A_c x, \tag{3}$$

where the vector $x(t) \in \mathbb{R}^n$ with $t \in \mathbb{R}$ and the matrix $A_c \in \mathbb{R}^{n\times n}$. For a given sampling time $h > 0$, a corresponding approximate discrete-time LTI system is given by

$$\Sigma_d : x(k+1) = A_d x(k), \tag{4}$$

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where k is an integer and A_d is obtained through an approximation of the matrix exponential $e^{A_c h}$. A square matrix A_c is said to be *Hurwitz stable* if all of its eigenvalues have negative real part, that is, they all lie in the open left half of the complex plane. A square matrix A_d is said to be *Schur stable* if all its eigenvalues have magnitude less than one, that is, they all lie in the interior of the unit disc. The Hurwitz stability of a continuous-time system or the Schur stability of a discrete-time system is equivalent to *global uniform exponential stability* of the system.

3. *Lyapunov matrices*: A matrix P is a *Lyapunov matrix* for a matrix A_c if $P^* = P > 0$ and $A_c^*P + PA_c < 0$. In this case, $V(x) = x^*Px$ is a *quadratic Lyapunov function (QLF)* for the continuous-time LTI system (3). A matrix P is a *Stein matrix* for a matrix A_d if $P^* = P > 0$ and $A_d^*PA_d - P < 0$. In this case, $V(x) = x^*Px$ is a QLF for the discrete-time LTI system (4). The existence of a quadratic Lyapunov function for a continuous-system or a discrete-time LTI system is equivalent to global exponential stability of the system.

4. *Quadratic stability*: Given a finite set of Hurwitz stable matrices $\mathscr{A}_c$, a matrix P is a *common Lyapunov matrix (CLM)* for $\mathscr{A}_c$ if P is a Lyapunov matrix for all A_c in $\mathscr{A}_c$. In this case, we say that the continuous-time switching system (1) is *quadratically stable (QS)* with the Lyapunov function $V(x) = x^*Px$ and V is a *common quadratic Lyapunov function (CQLF)* for $\mathscr{A}_c$. Given a finite set of Schur stable matrices $\mathscr{A}_d$, a matrix P is a *common Stein matrix (CSM)* for $\mathscr{A}_d$ if P is a Stein matrix for all A_d in $\mathscr{A}_d$. In this case, we say that the discrete-time switching system (2) is QS with the Lyapunov function $V(x) = x^*Px$ and V is a CQLF for $\mathscr{A}_d$. In both continuous-time and discrete-time, quadratic stability guarantees global uniform exponential stability of the switching system [10].

5. *A-stability*: [11–13] An approximation $f : \mathbb{C} \to \mathbb{C}$ for the exponential function e^z is said to be *A-stable* if it maps the open left half plane into the open unit disc, that is

$$|f(z)| < 1 \quad \text{for } z + z^* < 0. \tag{5}$$

Our primary interest in this note is when f is a rational function: namely, it can be expressed as $f(z) = n(z)/d(z)$ where n and d are polynomials. Functions of this form, when they operate on a matrix, map eigenvalues in the open left half plane to the interior of the unit disc, that is they preserve stability. Well known approximations to the matrix exponential are of this type, for example a large class of Padé approximations [12,14].

6. *Maximum modulus theorem*: If f is analytic and non-constant in a domain D, then $|f(z)|$ cannot obtain its maximum in D at an interior point of D. A consequence of this result is that *a non-constant rational function f satisfies condition (5) if and only if all its poles have a positive real part and*

$$|f(j\omega)| \le 1 \quad \text{for all real } \omega \tag{6}$$

where $j = \sqrt{-1}$.

To see this, suppose that all of the poles of f have a positive real part and condition (6) holds. This condition implies that f is proper and since f has no poles in the closed left-half plane, it is bounded in the closed left-half plane. Suppose $|f|$ has a maximum in the closed left-half plane. Since f is analytic in the open left-half plane, the maximum modulus theorem tells us that the maximum cannot occur in the open left-half plane, so the maximum must occur at $j\omega_0$ for some real ω_0 and whenever $\Re(z) < 0$, we have $|f(z)| < |f(j\omega_0)| \le 1$. Hence $|f(z)| < 1$ whenever $z + z^* < 0$. If $|f|$ does not have a maximum in the closed left-half plane, then it must have a supremum and

$$\sup_{\Re(z)\le 0} |f(z)| = |f(\infty)| = \sup_{\omega\in\mathbb{R}} |f(j\omega)| \le 1.$$

In this case, (5) also holds. Now suppose that (5) holds. This condition implies that $|f(z)| \le 1$ when $\Re(z) \le 0$ which in turns implies that f has no poles in the closed left-half plane. Hence, all poles of f must be in open right-half plane. Also, condition (6) holds.

3. Main result

We now present the main result of the paper.

Theorem 1. *Suppose that f is a rational function and*

$$|f(z)| < 1 \quad \text{when } z + z^* < 0. \tag{7}$$

Then, for any square matrix A,

$$\|f(A)\| < 1 \quad \text{when } A + A^* < 0 \tag{8}$$

where $\|f(A)\|$ denotes the maximum singular value of $f(A)$.

Proof. When $A + A^* < 0$ there exists $\alpha > 0$ such that

$$A + A^* \le -2\alpha I. \tag{9}$$

To prove our desired result we first note that

$$A = B(-1) \tag{10}$$

where B is the matrix valued function of a complex variable z defined by

$$B(z) = -\frac{z}{2}(A + A^* + 2\alpha I) + \frac{1}{2}(A - A^*) - \alpha I. \tag{11}$$

Now observe that

$$B(z) + B(z)^* = -\Re(z)(A + A^* + 2\alpha I) - 2\alpha I \tag{12}$$

and, using (9), we see that, whenever $\Re(z) \le 0$,

$$B(z) + B(z)^* \le -2\alpha I < 0.$$

Thus, all the eigenvalues of $B(z)$ have a negative real part when $\Re(z) \le 0$.

Consider now any real ω and note that

$$B(j\omega) = -\alpha I + jC$$

where C is the Hermitian matrix $-\frac{\omega}{2}(A + A^* + 2\alpha I) - \frac{j}{2}(A - A^*)$. Since C is Hermitian, there exists a unitary matrix[1] U such that $C = U\Omega U^*$ where Ω is diagonal with real diagonal elements $\omega_1, \ldots, \omega_n$. Hence

$$B(j\omega) = U(-\alpha I + j\Omega)U^*.$$

Also

$$f(B(j\omega)) = f\left(U(-\alpha I + j\Omega)U^*\right) = Uf(-\alpha I + j\Omega)U^*.$$

Since U is unitary, we have

$$\|f(B(j\omega))\| = \|Uf(-\alpha I + j\Omega)U^*\| = \|f(-\alpha I + j\Omega)\|.$$

Since the matrix $f(-\alpha I + j\Omega)$ is diagonal with diagonal elements $f(-\alpha + j\omega_i),\ i = 1, \ldots, n$, we must have

$$\|f(-\alpha I + j\Omega)\| = \max_{i=1,\ldots,n} |f(-\alpha + j\omega_i)|.$$

It follows from hypotheses (7) on f that $|f(-\alpha + j\omega_i)| < 1$ for $i = 1, \ldots, n$. Hence,

$$\|f(B(j\omega))\| < 1. \tag{13}$$

Since $\|f(A)\|$ is the maximum singular value of $f(A)$, there are two vectors u and v of norm one such that

$$\|f(A)\| = u^*f(A)v. \tag{14}$$

Defining the rational function ϕ by

$$\phi(z) = u^*f(B(z))v$$

[1] U is unitary if $U^*U = UU^* = I$.