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Pole placement by parametric output feedback

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ABSTRACT

This note presents a new analytical solution to the problem of pole placement via constant output feedback under the condition $m + p \geq n$, where n, m, and p are the number of states, inputs and outputs, respectively. The approach is based upon parametric eigenstructure assignment of linear time-invariant multivariable systems in combination with a special explicit formulation of the pole assignment equations. Thus, the resulting analytical solution explicitly offers all remaining $mp - n$ degrees of freedom beyond eigenvalue assignment which can be used for additional design goals such as response shaping, minimizing the norm of the feedback matrix, and robust control, respectively.

1. Introduction

Besides optimal control the pole placement approach is one of the most popular design methods in linear control theory. While in the case of complete state feedback the problem of finding a constant feedback matrix which assigns an arbitrary selected set of self-conjugate complex numbers as spectrum of the closed-loop system is completely solved (see e.g. [1]) to deal with static output feedback is much harder. This is mainly due to the fact that in multi-input–multi-output (MIMO) systems pole placement is a nonlinear problem and demands solving a set of nonlinear algebraic equations in the unknown gain parameters whose solution may not exist in the case of output feedback. However, for controllable systems and state feedback these equations always have a solution [1] and in MIMO systems this solution is even not unique. In this case, there are additional degrees-of-freedom (dof) beyond pole placement which can be used for further design goals such as eigenvector or eigenstructure assignment.

Based on the results in [2] on eigenstructure assignment in the case of state feedback several solutions to the problem of pole placement by static output feedback have been reported during the last three decades [3–10] and just recently another new approach has been presented in [11,12]. Most of them rely on the fundamental result of [13,14], which is also known as Kimura's condition, that for the generic system all closed-loop poles can be assigned almost arbitrary if $m+p \geq n+1$ where n, m, p denote the system order and the number of inputs and outputs, respectively. On the other hand in [15] it was shown that a necessary and sufficient condition for arbitrary pole assignment for the generic system is $mp \geq n$ if complex feedback gains are allowed and finally in [16,17] Wang presented the important result that for real static output feedback $mp > n$ is sufficient for generic pole assignability. However, up to now there is no closed-form solution to the problem of finding a real feedback under Wang's condition which is certainly due to the fact that in general pole placement via static output feedback is $\mathcal{NP}$-hard [18]. Moreover, if $mp > n$ one is interested in a solution which encompasses all remaining $mp - n$ dof beyond pole placement. Meanwhile, under Kimura's condition there exist several design techniques (see e.g. [6,7,10]) which offer such a parametric solution and in [11] the authors presented a new noniterative approach to pole placement based on eigenstructure assignment which improves Kimura's sufficient condition to $m + p \geq n$ while in [12] this result was extended to even encompass some cases for which $m + p < n < mp$. A general overview on static output feedback which covers several different design techniques can be found in [19].

In this note, we seize the suggestions from [10,11] and by combining them with the results from [20] we are able to develop a straightforward noniterative procedure for pole placement by parametric output feedback. In Section 2, after statement of the problem the fundamental properties of eigenstructure assignment are shortly reviewed and for $m = p = 2$ an analytical expression for the direct solution of the pole assignment equation is developed. Based on these preliminary results a closed-form parametric solution to the problem of pole placement by constant output feedback under the condition $m + p \geq n$ is presented in Section 3 while Section 4 gives a numerical example before the main results of this note are summarized in Section 5.

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2. Problem statement and preliminaries

In this section, the problem of pole placement by constant output feedback is stated and some fundamental results on eigenstructure assignment as well as a formulation of the problem based on an explicit expression of the closed loop characteristic polynomial are shortly reviewed.

Consider the completely controllable and observable linear time-invariant multivariable system

$$\dot{x} = Ax + Bu, \qquad y = Cx \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$. The real constant matrices A, B, and C are of appropriate dimensions and it is assumed that $\operatorname{rank}(B) = m \geq 2$ and $\operatorname{rank}(C) = p \geq 2$. If the real constant output feedback

$$u = Ky \tag{2}$$

is applied to (1) the closed-loop state equation becomes

$$\dot{x} = (A + BKC)x = A_c x. \tag{3}$$

The problem of pole placement or eigenvalue assignment by constant output feedback then means to find the real matrix K in (2) such that the spectrum $\sigma\{A_c\}$ of A_c coincides with the given set $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of self-conjugate complex values.

2.1. Eigenstructure assignment

In the context of eigenstructure assignment by constant output feedback the set $\Lambda = \{\Lambda_1, \Lambda_2\}$ of closed-loop eigenvalues is usually divided into two self-conjugate sets of arbitrarily selected distinct complex numbers $\Lambda_1 = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ and $\Lambda_2 = \{\lambda_{r+1}, \lambda_{r+2}, \ldots, \lambda_n\}$. Then in a first step the set Λ_1 is associated with the spectrum of A_c and the closed-loop eigenvectors v_i via

$$A_c v_i = (A + BKC)v_i = \lambda_i v_i, \quad i = 1, \ldots, r \tag{4}$$

which can also be written as

$$[A - \lambda_i I, B]\begin{bmatrix} v_i \\ KCv_i \end{bmatrix} = 0, \quad i = 1, \ldots, r. \tag{5}$$

If (A, B) is completely controllable $\operatorname{rank}[A - \lambda_i I, B] = n$, $\forall \lambda_i \in \mathbb{C}$ [21] and thus the right nullspace

$$\ker\{[A - \lambda_i I, B]\} = \ker\{S_i\} = \begin{bmatrix} N_i \\ M_i \end{bmatrix} \tag{6}$$

is of dimension m. Then the closed-loop eigenvectors v_i and the input directions

$$h_i = KCv_i \tag{7}$$

can be parameterized by nonzero parameter vectors $q_i \in \mathbb{C}^m$ (see e.g. [2,10,11])

$$v_i = N_i q_i \tag{8}$$

$$h_i = M_i q_i. \tag{9}$$

Remark 1. Since the columns of $\ker\{S_i\}$ can arbitrarily be scaled by any nonzero scalar the eigenvectors v_i and input directions h_i in (8), (9) are only determined except for their length and so are the parameter vectors q_i. Thus, each parameter vector only provides $(m-1)$ dof to assign the corresponding eigenvector v_i within the m-dimensional subspace of $\mathbb{C}^n$ spanned by the columns of N_i [2]. Moreover, it directly follows from elementary matrix theory that for a self-conjugate complex pair $\lambda_{i1} = \lambda_{i2}^*$ in the set Λ_1 we also have $v_{i1} = v_{i2}^*$, $N_{i1} = N_{i2}^*$ and this implies $q_{i1} = q_{i2}^*$. Therefore, the parameter vectors q_i associated with the set Λ_1 are not completely free but must also constitute a self-conjugate set. However, this does not reduce the available dof provided by the set q_i, $i = 1, \ldots, r$ since a complex q_i offers $m-1$ complex and $2(m-1)$ real dof, respectively.

Now we come back to (8), (9) and substitute them into (7) to get the homogeneous equation

$$(M_i - KCN_i)q_i = 0 \tag{10}$$

which has a nonzero solution $q_i \neq 0$ iff

$$\det(M_i - KCN_i) = 0. \tag{11}$$

Obviously, (11) can be used to assign λ_i as closed-loop pole [22] while the parameter vector q_i associated with λ_i via (10) then explicitly offers $(m-1)$ additional dof beyond eigenvalue assignment. Thus, to assign the r numbers in Λ_1 as closed-loop eigenvalues K must solve the linear equation

$$K(CV_r) = \tilde{Q}_r \tag{12}$$

where the $q_i \neq 0$, $i = 1, \ldots, r$ are considered as free parameters and

$$V_r = [N_1 q_1, \ldots, N_r q_r] \tag{13}$$

$$\tilde{Q}_r = [M_1 q_1, \ldots, M_r q_r]. \tag{14}$$

Obviously, a solution of (12) exists for almost any choice of the set Λ_1 and $q_i \neq 0$, $i = 1, \ldots, r$ if the condition $\operatorname{rank}(CV_r) = r$ holds which in turn implies $r \leq p$. For $r = p$, the usual choice in the literature on eigenstructure assignment, the solution $K = \tilde{Q}_r(CV_r)^{-1}$ of (12) explicitly exhibits all mp dof provided by $K \in \mathbb{R}^{m\times p}$ in the shape of the p eigenvalues from Λ_1 and the p corresponding parameter vectors $q_i \neq 0$, $i = 1, \ldots, p$. Thus, to assign the remaining $n-p$ eigenvalues in Λ_2 the parameter vectors $q_i \neq 0$, $i = 1, \ldots, p$ are not arbitrary but must undergo some restrictions. In the following this can be seen if all investigations carried out so far with right eigenvectors v_i, input directions h_i and (right) parameter vectors q_i are accomplished with left eigenvectors w_j', output directions l_j' and corresponding (left) parameter vectors z_j' for the eigenvalues in Λ_2 where the prime denotes transpose. To this end instead of (4) we start with the relation

$$w_j' A_c = w_j'(A + BKC) = \lambda_j w_j', \quad j = r+1, \ldots, n \tag{15}$$

or

$$[w_j', w_j' BK]\begin{bmatrix} A - \lambda_j I \\ C \end{bmatrix} = 0', \quad j = r+1, \ldots, n \tag{16}$$

where

$$\operatorname{rank}\begin{bmatrix} A - \lambda_j I \\ C \end{bmatrix} = \operatorname{rank}(T_j) = n, \quad \forall \lambda_j \in \mathbb{C}$$

if (A, C) is an observable pair [21]. Therefore, the p-dimensional left nullspace of T_j can be calculated from

$$[D_j, E_j] \cdot T_j = 0 \tag{17}$$

and the closed-loop left eigenvectors w_j' and output directions

$$l_j' = w_j' BK \tag{18}$$

are parameterized by nonzero left parameter vectors $z_j' \in \mathbb{C}^p$

$$w_j' = z_j' D_j \tag{19}$$

$$l_j' = z_j' E_j. \tag{20}$$

Finally, from (18)–(20) we get the dual version of (10)

$$z_j'(E_j - D_j BK) = 0' \tag{21}$$

with the corresponding necessary condition

$$\det(E_j - D_j BK) = 0 \tag{22}$$

and

$$(W_{n-r}' B)K = \tilde{Z}_{n-r}' \tag{23}$$