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Local asymptotic feedback stabilization to a submanifold: Topological conditions

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Abstract

We consider the problem of local asymptotic feedback stabilization of a control system $\dot{x} = f(x, u)$ defined in $\mathbb{R}^n$ to a compact, connected, oriented, embedded codimension one submanifold P of $\mathbb{R}^n$ using a continuous feedback law. This generalizes the problem of local asymptotic feedback stabilization to a point which has been previously considered in the control theory literature. It is natural to expect the topology of P to play a role in deciding whether or not local asymptotic stabilization to P of the system $\dot{x} = f(x, u)$ is feasible via continuous feedback, and our aim in this paper is precisely to outline necessary conditions on the topology of P for stabilization via continuous feedback to be achievable.

Keywords: Feedback stabilization; Homology theory; Degree theory

1. Introduction

Consider Brockett's non-holonomic integrator, given by

$$\begin{cases} \dot{x} = u, \\ \dot{y} = v, \\ \dot{z} = xv - yu, \end{cases}$$

where $(x, y, z) \in \mathbb{R}^3$ and $(u, v) \in \mathbb{R}^2$. It is well known [1,2] that there exists no continuous feedback law that asymptotically stabilizes this system to the origin of $\mathbb{R}^3$. On the other hand, the continuous feedback law given by

$$(x, y, z) \mapsto (u, v) = (-y - x(x^2 + y^2 - 1), \\ x - y(x^2 + y^2 - 1)) \quad (1)$$

locally asymptotically stabilizes Brockett's non-holonomic integrator to the unit cylinder $\{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 = 1\}$ of $\mathbb{R}^3$, as is seen for example by choosing $(x, y, z) \mapsto V(x, y, z) = (x^2 + y^2 - 1)^2$ as a Lyapunov function. In a similar vein, there exists no continuous feedback law that locally asymptotically stabilizes the system

$$\begin{cases} \dot{x} = u, \\ \dot{y} = v, \\ \dot{z} = (yu - xv)e^z, \end{cases}$$

to the origin of $\mathbb{R}^3$; yet, the feedback law given in Eq. (1) locally asymptotically stabilizes this system to the unit circle $\{(x, y, 0) \in \mathbb{R}^3 | x^2 + y^2 = 1\}$ in the $x - y$-plane.

These observations naturally lead to the following question: Assume a given system in $\mathbb{R}^n$ cannot be locally asymptotically stabilized to any point using a continuous feedback law; is it possible nevertheless to find a continuous feedback law that does locally asymptotically stabilize it to some other subset of $\mathbb{R}^n$? In particular, is it possible to find a continuous feedback law that locally asymptotically stabilizes the system to a given submanifold of $\mathbb{R}^n$? The motivation for studying this problem arises from the fact that stabilization to a submanifold can be considered as the next best thing for a system that cannot be stabilized to a point. In other words, if a system cannot be brought to equilibrium using continuous feedback, it may be possible nevertheless to have it exhibit some other behavior of interest. In the case of Brockett's non-holonomic integrator, for example, knowing that local asymptotic stabilization to a point via continuous feedback is not feasible, one could ask

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whether stabilization to a sphere in $\mathbb{R}^3$ instead is feasible. We shall see in this paper that the topology of the sphere disallows the existence of such a feedback law.

Let Ω be an open subset of $\mathbb{R}^n \times \mathbb{R}^m$, and let $f\colon \Omega \to \mathbb{R}^n$ be a continuous map. Let $x \mapsto u(x) \in \mathbb{R}^m$ be a continuous feedback law. Let P be a compact p-dimensional embedded submanifold of $\mathbb{R}^n$, invariant for the system $\dot{x} = f(x, u(x))$, that is all trajectories of this ordinary differential equation with initial condition in P remain in P. f defines the open-loop control system, whereas g defines the closed-loop control system. We start with the following definitions, borrowed from [7]:

Definition 1. The invariant submanifold P is said to be an asymptotically stable submanifold for the pair (f, u) if for every open neighborhood U of P, there exists an open neighborhood $U' \subset U$ of P such that every trajectory $t \mapsto x(t)$ of the system $\dot{x}(t) = f(x(t), u(x(t)))$ with initial point $x \in U'$ remains in U for all time and such that $d(x(t), P) \to 0$ as $t \to \infty$ (where d denotes Euclidean distance in $\mathbb{R}^n$).

Definition 2. A C^∞ function $V\colon U \mapsto \mathbb{R}$ on an open neighborhood $U \subset \mathbb{R}^n$ of P that satisfies

1. $V(x) \geqslant 0 \forall x \in U$ and $V(x) = 0$ if and only if $x \in P$,
2. $\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) < 0$ on $U \backslash P$,
3. V tends to a constant (possibly infinite) value on the boundary ∂U of U in $\mathbb{R}^n$,

will be called a Lyapunov function for the triple (f, u, P).

We shall make use of the following notion and result from differential topology. We refer the reader to [4] for more general statements.

Definition 3. A tubular neighborhood of P in $\mathbb{R}^n$ is a pair (ϕ, ζ), where $\zeta = (\pi, E, P)$ is a real vector bundle over P and $\phi\colon E \to \mathbb{R}^n$ is an embedding such that:

1. $\phi|_P = 1_P$, where P is identified with the zero section of E.
2. $\phi(E)$ is an open neighborhood of P in $\mathbb{R}^n$.

Theorem 1. *Let $P \subset \mathbb{R}^n$ be an embedded submanifold. Then P has a tubular neighborhood in $\mathbb{R}^n$.*

Remark. If, in addition, P is assumed to be oriented and of codimension one, then the normal bundle of P in $\mathbb{R}^n$ is trivial, and in that case we can assume with no loss of generality that the vector bundle E of Definition 3 is the trivial bundle $E = P \times \mathbb{R}$.

We shall make use of the following results of [7].

Theorem 2. *P is an asymptotically stable submanifold of the pair (f, u) if and only if there exists a Lyapunov function for the triple (f, u, P) defined on an open neighborhood of P.*

Theorem 3. *The level surfaces of a Lyapunov function for the triple (f, u, P) are homotopically equivalent to the boundary of a closed tubular neighborhood of P.*

Using these results, we shall state necessary conditions for local asymptotic stabilization to a submanifold P of $\mathbb{R}^n$. In all that follows, we shall make the following assumption:

A1: P is a compact, connected, oriented, embedded submanifold of $\mathbb{R}^n$ of codimension one.

Remark. Under Assumption A1, the normal bundle of P in $\mathbb{R}^n$ is trivial, and since the tangent bundle of $\mathbb{R}^n$ itself is trivial, the Stiefel–Whitney classes $\{w_i(TP)\}_{i=1}^{n-1}$ of the tangent bundle TP of P are zero. Hence, all the Stiefel–Whitney numbers of TP are zero as well. It then follows from Thom's theorem [5] that P is the boundary of an open relatively compact submanifold of $\mathbb{R}^n$.

Following assumption A1, for all $\delta > 0$, we shall denote by P_δ the image under the embedding ϕ of the open subset $P \times]-\delta, \delta[$ of E. We shall denote by ∂P_δ the boundary of P_δ in $\mathbb{R}^n$; since ϕ is an embedding, $\partial P_\delta = \partial^+ P_\delta \cup \partial^- P_\delta$, where $\partial^+ P_\delta = \phi(P \times \{-\delta\})$ and $\partial^- P_\delta = \phi(P \times \{\delta\})$; ∂P_δ is therefore the union of two disjoint submanifolds, each of which is diffeomorphic to P. Following the previous remark, both $\partial^+ P_\delta$ and $\partial^- P_\delta$ are boundaries of open submanifolds of $\mathbb{R}^n$; modifying ϕ if necessary, we shall assume $\partial^+ P_\delta$ is the "outer" boundary of P_δ, i.e. that $\partial^- P_\delta$ is included in the open submanifold of $\mathbb{R}^n$ of which $\partial^+ P_\delta$is the boundary.

2. Necessary conditions

Assume P is an asymptotically stable submanifold for the pair (f, u); it follows from Theorem 2 that there exists an open neighborhood U of P in $\mathbb{R}^n$ and a Lyapunov function $V\colon U \to \mathbb{R}$ for the triple (f, u, P) on U. Since P is assumed to be compact, we can, without loss of generality, choose U to be a relatively compact open neighborhood of P in $\mathbb{R}^n$. Furthermore, since $\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) < 0$ on $U \backslash P$, we have in particular that $f(x, u(x)) \neq 0 \ \forall x \in U \backslash P$. We have:

Lemma 1. $\forall c \in V(U \backslash P)$, $V^{-1}(c)$ *is an oriented submanifold of $\mathbb{R}^n$ with two connected components. Furthermore, restricting the open neighborhood U of P if necessary (and hence the domain of the Lyapunov function V) there exists $c' > 0$ small enough such that $\forall c \in]0, c'[$, $V^{-1}(c)$ is compact.*

Proof. The restriction of the mapping V to $U \backslash P$ has constant rank 1. Indeed, assume to the contrary, that at some point $p \in U \backslash P$, we have $\mathrm{d}V(p) = 0$. Since $p \notin P$, we have $V(p) > 0$. Then, $\frac{\partial V}{\partial x_i}(p) = 0$ for all $i = 1, \ldots, n$. Consider the trajectory $t \mapsto x(t)$ of the system $\dot{x}(t) = f(x(t), u(t))$ with initial condition $x(t_0) = p$. Then $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=t_0} V(x(t)) = 0$, contradicting assumption (2) in Definition 2. Hence, $\mathrm{d}V$ is non-zero at all points of $U \backslash P$, from which it follows that V has constant rank 1 on $U \backslash P$. It follows from the constant rank theorem that $\forall c \in V(U \backslash P)$, $V^{-1}(c)$ is a submanifold of $U \backslash P$, and since $U \backslash P$ is an open submanifold of $\mathbb{R}^n$, it follows that $V^{-1}(c)$ is a submanifold of $\mathbb{R}^n$; furthermore, since $\mathbb{R}^n$ is oriented, so is $V^{-1}(c)$. Since,