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Load balancing of dynamical distribution networks with flow constraints and unknown in/outflows

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ABSTRACT

We consider a basic model of a dynamical distribution network, modeled as a directed graph with storage variables corresponding to every vertex and flow inputs corresponding to every edge, subject to unknown but constant inflows and outflows. As a preparatory result it is shown how a distributed proportional–integral controller structure, associating with every edge of the graph a controller state, will regulate the state variables of the vertices, irrespective of the unknown constant inflows and outflows, in the sense that the storage variables converge to the same value (load balancing or consensus). This will be proved by identifying the closed-loop system as a port-Hamiltonian system, and modifying the Hamiltonian function into a Lyapunov function, dependent on the value of the vector of constant inflows and outflows. In the main part of the paper the same problem will be addressed for the case that the input flow variables are *constrained* to take value in an arbitrary interval. We will derive sufficient and necessary conditions for load balancing, which only depend on the structure of the network in relation with the flow constraints.

1. Introduction

In this paper, we study a basic model for the dynamics of a distribution network. Identifying the network with a directed graph we associate with every vertex of the graph a state variable corresponding to *storage*, and with every edge a control input variable corresponding to *flow*, which is constrained to take value in a given closed interval. Furthermore, some of the vertices serve as terminals where an unknown but constant flow may enter or leave the network in such a way that the total sum of inflows and outflows is equal to zero. The control problem to be studied is to derive necessary and sufficient conditions for a distributed control structure (the control input corresponding to a given edge only depending on the difference of the state variables of the adjacent vertices) which will ensure that the state variables associated to all vertices will converge to the same value equal to the average of the initial condition, irrespective of the values of the constant unknown inflows and outflows.

The structure of the paper is as follows. Some preliminaries and notations will be given in Section 2. In Section 3 we will show how in the absence of constraints on the flow input variables a distributed proportional–integral (PI) controller structure, associating with every edge of the graph a controller state, will solve the problem if and only if the graph is weakly connected. This will be shown by identifying the closed-loop system as a port-Hamiltonian system, with state variables associated both to the vertices and the edges of the graph, in line with the general definition of port-Hamiltonian systems on graphs [1–4]; see also [5,6]. The proof of asymptotic load balancing will be given by modifying, depending on the vector of constant inflows and outflows, the total Hamiltonian function into a Lyapunov function. In the examples the obtained PI-controller often has a clear physical interpretation, emulating the physical action of adding energy storage and damping to the edges.

The main contribution of the paper resides in Sections 4 and 5, where the same problem is addressed for the case of *constraints* on the flow input variables. In Section 4 it will be shown that in the case of *zero* inflow and outflow the state variables of the vertices converge to the same value if and only if the network is strongly connected. This will be shown by constructing a C^1 Lyapunov function based on the total Hamiltonian and the constraint values. This same construction will be extended in Section 5 to the case of nonzero inflows and outflows, leading to the result that in this case asymptotic load balancing is reached if and only the graph is not only strongly connected but also *balanced*. Finally, Section 6 contains the conclusions.

Some preliminary results, in particular concerning Section 3, have been already reported before in [7].

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2. Preliminaries and notations

First we recall some standard definitions regarding directed graphs, as can be found e.g. in [8]. A *directed graph* $\mathcal{G}$ consists of a finite set $\mathcal{V}$ of *vertices* and a finite set $\mathcal{E}$ of *edges*, together with a mapping from $\mathcal{E}$ to the set of ordered pairs of $\mathcal{V}$, where no self-loops are allowed. Thus to any edge $e \in \mathcal{E}$ there corresponds an ordered pair $(v, w) \in \mathcal{V} \times \mathcal{V}$ (with $v \neq w$), representing the tail vertex v and the head vertex w of this edge.

A directed graph is completely specified by its *incidence matrix* B, which is an $n \times m$ matrix, n being the number of vertices and m being the number of edges, with (i, j)th element equal to 1 if the jth edge is towards vertex i, and equal to -1 if the jth edge is originating from vertex i, and 0 otherwise. Since we will only consider the directed graphs in this paper 'graph' will throughout mean 'directed graph' in the sequel. A directed graph is *strongly connected* if it is possible to reach any vertex starting from any other vertex by traversing edges following their directions. A directed graph is called *weakly connected* if it is possible to reach any vertex from every other vertex using the edges *not* taking into account their direction. A graph is weakly connected if and only if $\ker B^T = \operatorname{span} \mathbb{1}_n$. Here $\mathbb{1}_n$ denotes the n-dimensional vector with all elements equal to 1. A graph that is not weakly connected falls apart into a number of weakly connected subgraphs, called the connected components. The number of connected components is equal to $\dim \ker B^T$. For each vertex, the number of incoming edges is called the *in-degree* of the vertex and the number of outgoing edges its *out-degree*. A graph is called *balanced* if for every vertex their in-degree and out-degree of every vertex are equal. A graph is balanced if and only if $\mathbb{1}_n \in \ker B$.

Given a graph, we define its *vertex space* as the vector space of all functions from $\mathcal{V}$ to some linear space $\mathcal{R}$. In the rest of this paper we will take for simplicity $\mathcal{R} = \mathbb{R}$, in which case the vertex space can be identified with $\mathbb{R}^n$. Similarly, we define its *edge space* as the vector space of all functions from $\mathcal{E}$ to $\mathcal{R} = \mathbb{R}$, which can be identified with $\mathbb{R}^m$. In this way, the incidence matrix B of the graph can be also regarded as the matrix representation of a linear map from the edge space $\mathbb{R}^m$ to the vertex space $\mathbb{R}^n$.

Notation: for $a, b \in \mathbb{R}^m$ the notation $a \leqslant b$ will denote elementwise inequality $a_i \leq b_i,\ i = 1, \ldots, m$. For $a_i < b_i,\ i = 1, \ldots, m$ the multidimensional saturation function $\operatorname{sat}(x; a, b) : \mathbb{R}^m \to \mathbb{R}^m$ is defined as

$$\operatorname{sat}(x; a, b)_i = \begin{cases} a_i & \text{if } x_i \leq a_i, \\ x_i & \text{if } a_i < x_i < b_i, \\ b_i & \text{if } x_i \geq b_i, \end{cases} \quad i = 1, \ldots, m. \tag{1}$$

3. A dynamic network model with PI controller

Let us consider the following dynamical system defined on the graph $\mathcal{G}$

$$\begin{aligned} \dot{x} &= Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \\ y &= B^T \frac{\partial H}{\partial x}(x), \quad y \in \mathbb{R}^m, \end{aligned} \tag{2}$$

where $H : \mathbb{R}^n \to \mathbb{R}$ is any differentiable function, and $\frac{\partial H}{\partial x}(x)$ denotes the column vector of partial derivatives of H. Here the ith element x_i of the state vector x is the state variable associated to the ith vertex, while u_j is a flow input variable associated to the jth edge of the graph. System (2) defines a port-Hamiltonian system [9,10], satisfying the energy-balance

$$\frac{d}{dt} H = u^T y. \tag{3}$$

Note that geometrically its state space is the vertex space, its input space is the edge space, while its output space is the dual of the edge space.

Example 3.1 (*Hydraulic Network*)**.** Consider a hydraulic network, modeled as a directed graph with vertices (nodes) corresponding to reservoirs, and edges (branches) corresponding to pipes. Let x_i be the amount of water stored at vertex i, and u_j the flow through edge j. Then the mass-balance of the network is summarized in

$$\dot{x} = Bu, \tag{4}$$

where B is the incidence matrix of the graph. Let furthermore $H(x)$ denote the stored energy in the reservoirs (e.g., gravitational energy). Then $P_i := \frac{\partial H}{\partial x_i}(x),\ i = 1, \ldots, n$, are the *pressures* at the vertices, and the output vector $y = B^T \frac{\partial H}{\partial x}(x)$ is the vector whose jth element is the pressure *difference* $P_i - P_k$ across the jth edge linking vertex k to vertex i.

As a next step we will extend the dynamical system (2) with a vector d of *inflows and outflows*

$$\begin{aligned} \dot{x} &= Bu + Ed, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m,\ d \in \mathbb{R}^k \\ y &= B^T \frac{\partial H}{\partial x}(x), \quad y \in \mathbb{R}^m, \end{aligned} \tag{5}$$

with E an $n \times k$ matrix whose columns consist of exactly one entry equal to 1 (inflow) or -1 (outflow), while the rest of the elements is zero. Thus E specifies the k terminal vertices where flows can enter or leave the network.

In this paper we will regard d as a vector of constant *disturbances*, and we want to investigate control schemes which ensure asymptotic load balancing of the state vector x irrespective of the (unknown) disturbance d. The simplest control possibility is to apply a proportional output feedback

$$u = -Ry = -RB^T \frac{\partial H}{\partial x}(x), \tag{6}$$

where R is a diagonal matrix with strictly positive diagonal elements $r_1, \ldots, r_m$. Note that this defines a *decentralized* control scheme if H is of the form $H(x) = H_1(x_1) + \cdots + H_n(x_n)$, in which case the ith input as given by (6) equals r_i times the difference of the component of $\frac{\partial H}{\partial x}(x)$ corresponding to the head vertex of the ith edge and the component of $\frac{\partial H}{\partial x}(x)$ corresponding to its tail vertex. This control scheme leads to the closed-loop system

$$\dot{x} = -BRB^T \frac{\partial H}{\partial x}(x) + Ed. \tag{7}$$

In the case of *zero* in/outflows $d = 0$ this implies the energy-balance

$$\frac{d}{dt} H = -\frac{\partial^T H}{\partial x}(x) BRB^T \frac{\partial H}{\partial x}(x) \leq 0. \tag{8}$$

Hence if H is radially unbounded it follows that the system trajectories of the closed-loop system (7) will converge to the set

$$\mathcal{E} := \left\{ x \mid B^T \frac{\partial H}{\partial x}(x) = 0 \right\} \tag{9}$$

and thus to the load balancing set

$$\mathcal{E} = \left\{ x \mid \frac{\partial H}{\partial x}(x) = \alpha \mathbb{1}, \alpha \in \mathbb{R} \right\}$$

if and only if $\ker B^T = \operatorname{span}\{\mathbb{1}\}$, or equivalently [8], if and only if the graph is *weakly connected*.

In particular, for the standard Hamiltonian $H(x) = \frac{1}{2}\|x\|^2$ this means that the state variables $x_i(t),\ i = 1, \ldots, n$, converge to a common value α as $t \to \infty$. Since $\frac{d}{dt} \mathbb{1}^T x(t) = 0$ it follows that this common value is given as $\alpha = \frac{1}{n} \sum_{i=1}^n x_i(0)$.

For $d \neq 0$ proportional control $u = -Ry$ will not be sufficient to reach load balancing, since the disturbance d can only be attenuated at the expense of increasing the gains in the matrix R. Hence