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On a class of positive linear differential equations with infinite delay

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ABSTRACT

We give an explicit criterion for positivity of the solution semigroup of linear differential equations with infinite delay and a Perron–Frobenius type theorem for positive equations. Furthermore, a novel criterion for the exponential asymptotic stability of positive equations is presented. Finally, we provide a sufficient condition for the exponential asymptotic stability of positive equations subjected to structured perturbations. A simple example is given to illustrate the obtained results.

1. Introduction

Delay differential equations have numerous applications in science and engineering. They are used as models for a variety of phenomena in life sciences, physics and technology, chemistry and economics; see e.g. [1,2].

Theory of differential equations with infinite delay was established and developed in the 1970s, and 1980s; see e.g. [3–6] and references therein. A comprehensive theory of differential equations with infinite delay can be found in [4]. In particular, problems of stability of linear differential equations with infinite delay have been studied in [7,5]. Recently, problems of differential equations with infinite delay have attracted much attention from researchers; see e.g. [8–11].

Although there are many works devoted to the study of stability of differential equations with infinite delay, however, to the best of our knowledge, *aspects of positivity in stability problems of linear differential equations with infinite delay have not yet exploited in the literature*. Roughly speaking, a dynamical system is called *positive* if for any nonnegative initial condition, the corresponding solution of the system is also nonnegative. Positive dynamical systems play an important role in the modelling of dynamical phenomena whose variables are restricted to be nonnegative. They are often encountered in applications, for example, networks of reservoirs, industrial processes involving chemical reactors, heat exchangers, distillation columns, storage systems, hierarchical systems, compartmental systems used for modelling transport and accumulation phenomena of substances; see e.g. [12,13]. The mathematical theory of positive linear systems is based on the theory of nonnegative matrices founded by Perron and Frobenius; see e.g. [12,13].

Motivated by many applications in various scientific areas, problems of positive systems have been studied intensively and extensively in the recent times; see e.g. [14–20] and references therein.

In the present paper, we deal with linear differential equations with infinite delay of the form

$$\frac{dx}{dt}(t) = Ax(t) + \int_{-\infty}^{0} B(\theta)x(t+\theta)d\theta \qquad t \geq 0. \tag{1}$$

Such an equation is called positive if, and only if, its solution semigroup is positive. We will give an explicit criterion and a Perron–Frobenius type theorem for positive equations. Furthermore, we present a novel criterion for the exponential asymptotic stability of positive equations and a sufficient condition for the exponential asymptotic stability of positive equations subjected to structured perturbations.

2. Preliminaries

In this section, we shall define some notations and recall some well-known results which will be used in what follows. Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ where $\mathbb{C}$ and $\mathbb{R}$ denote the sets of all complex and all real numbers, respectively. Define $\mathbb{R}_- := \{s \in \mathbb{R} : s \leq 0\}$ and $\mathbb{R}_+ := \{s \in \mathbb{R} : s \geq 0\}$. For given $\gamma \in \mathbb{R}$, let us denote $\mathbb{C}_\gamma := \{z \in \mathbb{C} : \Re z \geq \gamma\}$. For an integer $l, q \geq 1$, $\mathbb{K}^l$ denotes the l-dimensional vector space over $\mathbb{K}$ and $\mathbb{K}^{l\times q}$ stands for the set of all $l \times q$-matrices with entries in $\mathbb{K}$. Inequalities between real matrices or vectors will be understood componentwise, i.e. for two real matrices $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathbb{R}^{l\times q}$, we write $A \geq B$ iff $a_{ij} \geq b_{ij}$ for $i = 1, \ldots, l$, $j = 1, \ldots, q$. We denote by $\mathbb{R}_+^{l\times q}$ the set of all nonnegative matrices

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$A \geq 0$. Similar notations are adopted for vectors. For $x \in \mathbb{K}^n$ and $P \in \mathbb{K}^{l\times q}$ we define $|x| = (|x_i|)$ and $|P| = (|p_{ij}|)$. A norm $\|\cdot\|$ on $\mathbb{K}^n$ is said to be *monotonic* if $\|x\| \leq \|y\|$ whenever $x, y \in \mathbb{K}^n$, $|x| \leq |y|$. Every p-norm on $\mathbb{K}^n$ ($\|x\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$, $1 \leq p < \infty$ and $\|x\|_\infty = \max_{i=1,2,\ldots,n} |x_i|$), is monotonic. Throughout the paper, if otherwise not stated, the norm of a matrix $P \in \mathbb{K}^{l\times q}$ is understood as its operator norm associated with a given pair of monotonic vector norms on $\mathbb{K}^l$ and $\mathbb{K}^q$, that is $\|P\| = \max\{\|Py\| : \|y\| = 1\}$. Note that, one has

$$P \in \mathbb{K}^{l\times q},\ Q \in \mathbb{R}_+^{l\times q}, \quad |P| \leq Q \Rightarrow \|P\| \leq \|\,|P|\,\| \leq \|Q\|, \tag{2}$$

see, e.g. [21]. For any matrix $A \in \mathbb{K}^{n\times n}$, the *spectral abscissa* of A is denoted by $\mu(A) = \max\{\Re\lambda : \lambda \in \sigma(A)\}$, where $\sigma(A) := \{\lambda \in \mathbb{C} : \det(\lambda I_n - A) = 0\}$ is the spectrum of A. $A \in \mathbb{R}^{n\times n}$ is called a *Metzler matrix* if all off-diagonal elements of A are nonnegative.

We now summarize in the following theorem some properties of Metzler matrices which will be used in what follows.

Theorem 2.1 ([21]). *Suppose that $A \in \mathbb{R}^{n\times n}$ is a Metzler matrix. Then*

(i) *(Perron–Frobenius) $\mu(A)$ is an eigenvalue of A and there exists a nonnegative eigenvector $x \neq 0$ such that $Ax = \mu(A)x$.*
(ii) *Given $\alpha \in \mathbb{R}$, there exists a nonzero vector $x \geq 0$ such that $Ax \geq \alpha x$ if and only if $\mu(A) \geq \alpha$.*
(iii) *$(tI_n - A)^{-1}$ exists and is nonnegative if and only if $t > \mu(A)$.*
(iv) *Given $B \in \mathbb{R}_+^{n\times n}$, $C \in \mathbb{C}^{n\times n}$. Then*

$$|C| \leq B \Longrightarrow \mu(A + C) \leq \mu(A + B).$$

Let $\mathbb{K}^{m\times n}$ be endowed with the norm $\|\cdot\|$ and let J be an interval of $\mathbb{R}$. Denote $C(J, \mathbb{K}^{m\times n})$, the vector space of all continuous functions on J with values in $\mathbb{K}^{m\times n}$. In particular, $C([\alpha, \beta], \mathbb{K}^{m\times n})$ is a Banach space endowed with the norm $\|\varphi\| := \max_{\theta\in[\alpha,\beta]} \|\varphi(\theta)\|$. For a matrix function $\varphi(\cdot) : J \to \mathbb{R}^{m\times n}$, we say that $\varphi(\cdot)$ is nonnegative and denote it by $\varphi \geq 0$ if $\varphi(\theta) \geq 0$ for all $\theta \in J$. A matrix function $\eta(\cdot) : J \to \mathbb{R}^{m\times n}$ is called non-decreasing on J, if $\eta(\theta_2) \geq \eta(\theta_1)$ for $\theta_1, \theta_2 \in J, \theta_1 < \theta_2$.

To end this section, we give a brief introduction to C_0-semigroups of bounded linear operators on Banach spaces which is used in what follows. Let X be a Banach space and let I be the identity operator on X. A family $(T(t))_{t\geq 0}$ of bounded linear operators on X is called a strongly continuous semigroup (or C_0-semigroup) if it satisfies

$$T(0) = I; \qquad T(t + s) = T(t)T(s), \quad t, s \geq 0,$$

and is strongly continuous in the following sense: for every $x \in X$, the orbit maps $\xi_x : t \mapsto \xi_x(t) := T(t)x$, are continuous from $\mathbb{R}_+$ into X. Then the generator $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ of $(T(t))_{t\geq 0}$ is the operator

$$\mathcal{A}x := \lim_{h\downarrow 0} \frac{1}{h}(T(h)x - x),$$

defined for every x in its domain $D(\mathcal{A}) := \{x \in X : \lim_{h\downarrow 0} \frac{1}{h}(T(h)x - x) \text{ exists}\}$. Furthermore, $\sigma(\mathcal{A}) := \{\lambda \in \mathbb{C} : \lambda I - \mathcal{A} \text{ is not bijective}\}$ and $\rho(\mathcal{A}) := \mathbb{C} \setminus \sigma(\mathcal{A})$ is called the spectrum of $\mathcal{A}$ and the resolvent set of $\mathcal{A}$, respectively. In particular, $R(\lambda, \mathcal{A}) := (\lambda I - \mathcal{A})^{-1}$ is called the resolvent of $\mathcal{A}$ at $\lambda \in \rho(\mathcal{A})$. For further information, we refer to [22].

3. Characterizations of positive linear differential equations with infinite delay

3.1. An explicit criterion for positive equations

Consider a linear differential equation with infinite delay of the form (1), where $A \in \mathbb{R}^{n\times n}$ and $B(\cdot) \in C(\mathbb{R}_-, \mathbb{R}^{n\times n})$ are given.

Existence and uniqueness of solution to (1) can be found in [3,4]. It is important to note that unlike to the finite delay case, the initial data is always part of the solution of (1). So, there is not a time with immediate regularization, and some kind of regularity must be imposed from the beginning; see e.g. [3]. This leads us to work with a canonical phase space:

$$\mathcal{C}_\gamma := \{\varphi \in C(\mathbb{R}_-, \mathbb{R}^n) : \lim_{\theta\to -\infty} e^{\gamma\theta}\varphi(\theta) \in \mathbb{R}^n\}.$$

where $\gamma > 0$ is given; see e.g. [3,4]. Recall that $\mathcal{C}_\gamma$ is a Banach space with the norm $\|\varphi\| := \sup_{\theta\in\mathbb{R}_-} \|e^{\gamma\theta}\varphi(\theta)\|$ and $\mathcal{C}_\gamma$ encompasses the set of all bounded continuous functions on $\mathbb{R}_-$. Furthermore, for a given $\lambda \in \mathbb{C}$ and a given $x_0 \in \mathbb{R}^n$, denote $(\omega_\lambda \otimes x_0)(\theta) := e^{\lambda\theta}x_0,\ \theta \in \mathbb{R}_-$. Then $\omega_\lambda \otimes x_0 \in \mathcal{C}_\gamma$, for any $x_0 \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}$ with $\Re\lambda > -\gamma$. Throughout this section, we always assume that

$$\int_{-\infty}^{0} e^{-\gamma\theta}\|B(\theta)\|d\theta < +\infty. \tag{3}$$

Eq. (1) can be rewritten as

$$\frac{dx}{dt}(t) = Lx_t, \quad t \in \mathbb{R}_+,$$

where $L : \mathcal{C}_\gamma \to \mathbb{R}^n;\ \varphi \mapsto L\varphi := A\varphi(0) + \int_{-\infty}^{0} B(\theta)\varphi(\theta)d\theta$ and $x_t(\theta) := x(t + \theta), \theta \in \mathbb{R}_-$. It follows from (3) that L is a bounded linear operator and $\|L\| \leq \|A\| + \int_{-\infty}^{0} e^{-\gamma\theta}\|B(s)\|d\theta$. For (1), we consider an initial condition of the form

$$x(\theta) = \varphi(\theta), \quad \theta \in \mathbb{R}_-, \tag{4}$$

where $\varphi \in \mathcal{C}_\gamma$ is given.

Definition 3.1. A continuous function $x : \mathbb{R} \to \mathbb{R}^n$ is called a solution of the initial value problem (1), (4) if, and only if,

(i) x is continuously differentiable on $\mathbb{R}_+$ and satisfies (1) for all $t \in \mathbb{R}_+$ and
(ii) $x(\theta) = \varphi(\theta),\ \theta \in \mathbb{R}_-$.

For given $\varphi \in \mathcal{C}_\gamma$, the initial value problem (1)–(4) has a unique solution, denoted by $x(\cdot; \varphi)$, provided (3) holds; see e.g. [4]. Furthermore, one associates (1) with a semigroup of solution operator on $\mathcal{C}_\gamma$. The semigroup is strongly continuous (C_0-semigroup) and is given by translation along the solution of (1)–(4):

$$T(t)\varphi := x_t(\cdot; \varphi), \quad t \in \mathbb{R}_+, \tag{5}$$

where $x_t(\theta; \varphi) := x(t+\theta; \varphi), \theta \in \mathbb{R}_-$. For further information and details, we refer readers to [7,5]. The semigroup $(T(t))_{t\geq 0}$ is called the *solution semigroup* of (1). In particular, the solution semigroup of the equation $\frac{dx}{dt} = 0$, is denoted by $(S(t))_{t\geq 0}$.

Set

$$\mathcal{H}(\lambda) := \lambda I_n - A - \int_{-\infty}^{0} e^{\lambda\theta}B(\theta)d\theta. \tag{6}$$

Define $B_1(\tau) := B(-\tau),\ \tau \in \mathbb{R}_+$. Then $\mathcal{H}(\lambda) = \lambda I_n - A - \widehat{B_1}(\lambda)$, where $\widehat{B_1}(\lambda) := \int_0^\infty e^{-\lambda\tau}B_1(\tau)d\tau$, for appropriate $\lambda \in \mathbb{C}$, is the *Laplace transform* of B_1; see e.g. [23]. By (3), $\widehat{B_1}(\cdot)$ is well-defined in $\mathbb{C}_{-\gamma}$. The equation $\det \mathcal{H}(\lambda) = 0$ is called the *characteristic equation* of (1).

The following theorem is a particular case of Theorem 4.5 in [5].

Theorem 3.2 ([5]). *Let $\mathcal{A}$ and $\mathcal{B}$ be the generators of the semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ and let $R(\cdot; \mathcal{A})$ and $R(\cdot; \mathcal{B})$ be the resolvents of $\mathcal{A}$ and $\mathcal{B}$, respectively. Then*

$$R(\lambda; \mathcal{A})\varphi = \omega_\lambda \otimes \mathcal{H}(\lambda)^{-1}\left(A(R(\lambda; \mathcal{B})\varphi)(0) + \int_{-\infty}^{0} B(\theta)R(\lambda; \mathcal{B})\varphi(\theta)d\theta\right) + R(\lambda; \mathcal{B})\varphi, \tag{7}$$

where $\mathcal{H}(\cdot)$ is given by (6) and $\lambda \in \mathbb{C}$ with $\Re\lambda$ sufficiently large.