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A converse Lyapunov theorem for almost sure stabilizability☆

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Abstract

We prove a converse Lyapunov theorem for almost sure stabilizability and almost sure asymptotic stabilizability of controlled diffusions: given a stochastic system a.s. stochastic open-loop stabilizable at the origin, we construct a lower semicontinuous positive definite function whose level sets form a local basis of viable neighborhoods of the equilibrium. This result provides, with the direct Lyapunov theorems proved in a companion paper, a complete Lyapunov-like characterization of the a.s. stabilizability.

Keywords: Degenerate diffusion; Almost sure stability; Stabilizability; Asymptotic stability; Stochastic control; Control Lyapunov function; Viscosity solutions; Hamilton–Jacobi–Bellman inequalities; Viability

1. Introduction

In this paper we provide a Lyapunov characterization of almost sure stochastic open loop stability at an equilibrium of controlled diffusion processes in $\mathbb{R}^N$

$$\text{(CSDE)}\begin{cases} \mathrm{d}X_t = f(X_t,\alpha_t)\,\mathrm{d}t + \sigma(X_t,\alpha_t)\,\mathrm{d}B_t, & \alpha_t \in A,\ t>0,\\ X_0 = x, \end{cases}$$

This notion has been introduced in a companion paper [6] by Bardi and the author (see also [5]): we say that (CSDE) is *a.s. (open-loop) stabilizable* if for any $\eta>0$ there exists $\delta>0$ such that, for any x with $|x|\leqslant\delta$, there exists α such that the corresponding process satisfies $|X_t|\leqslant\eta$ for all $t\geqslant 0$ almost surely. If, in addition, the trajectory is asymptotically approaching a.s. the equilibrium, we say the system is *a.s. (open-loop) asymptotically stabilizable*. The definitions imply in particular that these properties are never verified by nondegenerate processes. This stochastic stability describes a behavior very similar to a stable deterministic system and is stronger than pathwise stability and stability in probability (see [17,19,21]). We characterize it by means of appropriate control *Lyapunov functions*. These functions have been introduced in [5] and are lower semicontinuous (LSC), continuous at the equilibrium, positive definite, proper. Moreover, they satisfy the following *infinitesimal decrease condition*:

$$\sup_{\sigma(x,\alpha)^{\mathrm{T}}DV(x)=0} \{-DV(x)\cdot f(x,\alpha) - \mathrm{trace}[a(x,\alpha)D^2V(x)]\} \geqslant l(x), \tag{1}$$

where $a := \sigma\sigma^{\mathrm{T}}/2$, $l\equiv 0$ for Lyapunov functions and positive definite for strict Lyapunov functions. This is not a standard Hamilton–Jacobi–Bellman inequality, because the constraint on the controls depends on V: we are allowing diffusion only in the directions tangential to the sublevel sets of V. If we eliminate this constraint, the differential inequality which is left is the infinitesimal decrease condition on Lyapunov functions for the stability in probability (see [11,12]). We prove that V satisfies (1) if and only if it satisfies the following *monotonicity condition*:

$$\forall x\ \exists\alpha : \sup_{t\geqslant 0}\ \operatorname{ess\,sup}_{\omega\in\Omega}\left(V(X_t)+\int_0^t l(X_s)\,\mathrm{d}s\right)\leqslant V(x), \tag{2}$$

where the essential supremum is intended with respect to the probability measure $\mathbf{P}_x$. This means in particular that the process $V(X_t)$ is a positive *supermaxingale* according to the definition given in [10]: this is the natural counterpart of the requirement on the process $V(X_t)$ to be a

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positive supermartingale in the context of stability in probability. The monotonicity condition says that the sublevel sets $K_\mu := \{x\ V(x) \leqslant \mu\}$ are *viable* (*or weakly invariant*) with respect to (CSDE) in the sense that $\forall x \in K_\mu \exists \alpha$ such that $X_t \in K_\mu$ forever almost surely. One of the main tool used in this paper is the geometric Nagumo-type characterization of viability proved recently by Bardi and Jensen in [9] (see also [8,2] and the references therein for earlier related results).

In [6], Bardi and the author show that the existence of a Lyapunov function (respectively, of a strict Lyapunov function) implies the a.s. stabilizability (respectively, the a.s. asymptotic stabilizability) of the system to the equilibrium. As a simple example of application of this theory, we consider a radial function $V(x) = v(|x|)$, for some real smooth function v with $v'(r) > 0$ for $r > 0$. The system (CSDE) admits V as Lyapunov function if

$$\forall x\ \exists \alpha : \sigma(x, \alpha) \cdot x = 0 \quad f(x, \alpha) \cdot x + \operatorname{trace} a(x, \alpha) \leqslant 0.$$

Therefore, the following conditions are sufficient for the a.s. stabilizability: the radial component of the diffusion is null and its rotational component, which still plays a destabilizing role since trace$a(x, \alpha) \geqslant 0$, must be compensated by a negative radial component of f.

In this paper we prove that the existence of a Lyapunov function is also a necessary condition for a.s. stabilizability (and also for a.s. Lagrange stabilizability in the global case). A Lyapunov function for the system can be defined as

$$V(x) := \inf\{r | \exists \overline{\alpha} \text{ admissible control such that } |\overline{X}_t| \leqslant r \text{ almost surely } \forall t \geqslant 0\},$$

or equivalently as

$$V(x) := \inf_{\alpha \in \mathscr{A}_x} \sup_{t \geqslant 0} \operatorname{ess\,sup}_{\omega \in \Omega} |X_t^\alpha|.$$

We prove that this function is LSC, continuous at the origin, positive definite, proper, and satisfies the infinitesimal decrease condition (1) with $l \equiv 0$. For an a.s. asymptotic stabilizable systems in a bounded set $\mathcal{O}$, we build a positive definite Lipschitz continuous function l, related to the rate of decrease of the stable trajectories to the equilibrium, by the formula

$$V(x) := \inf_{\alpha \in \mathscr{A}_x} \operatorname{ess\,sup}_{\omega \in \Omega} \int_0^{+\infty} l(X_t^\alpha)\, \mathrm{d}t.$$

We show that V is finite, LSC, continuous at the origin, positive definite, proper, and satisfies the infinitesimal decrease condition (1).

In both cases the Lyapunov functions are *worst-case value function* of an appropriate stochastic optimal control problem: we minimize the worst possible cost over all possible paths. This is quite natural since these functions characterize a very strong stability notion. The link between worst-case value functions and viscosity solutions to geometric second order partial differential equations has been recently treated by Soner and Touzi in [24] (see also [10]). They considered stochastic target problems where the controller tries to steer almost surely a controlled process into a given target by judicial choices of controls. The interest in this kind of stochastic control problems in the *almost sure* setting comes from the relationship with mean curvature type geometric flows and from the applications to the super-replication problems in financial mathematics. Moreover, the a.s. stability of control systems affected by disturbances modelled as M-dimensional white noise is related to the so-called *worst-case stability (or robust stability)* of deterministic control systems affected by disturbances modelled as (deterministic) L^∞ functions with values in $\mathbb{R}^M$ (see [16]). The Lyapunov characterization of these two stability properties seems to be an useful tool to give a precise proof of this relationship (see [11]), while a direct proof based on the estimates among the trajectories of the two systems should be rather hard. An approach of this type has been used recently by Da Prato and Frankowska in [14] to prove the equivalence between the invariance with respect to a controlled stochastic system and the invariance with respect to a deterministic system with two (non competitive) controls.

We conclude with some additional references on converse Lyapunov theorems. For controlled deterministic systems, there are theorems characterizing the stochastic open-loop stabilizability by means of LSC appropriate Lyapunov functions (see [3]). Soravia in [28] showed that the stability at an equilibrium is equivalent to the continuity at such point of the value function $V(x) = \inf_{\alpha \in \mathscr{A}_x} \sup_{t \geqslant 0} U(X_t)$ where the level sets of U form a local basis of neighborhoods of the equilibrium. For asymptotically controllable systems, Sontag and Sussmann [26,27] provided a characterization of asymptotic controllability by means of continuous Lyapunov functions such as $V(x) = \inf_\alpha \int_0^{+\infty} l(X_t)\, \mathrm{d}t$ where l is an appropriate positive definite function. Recently Rifford [23] proved a converse Lyapunov theorem in the framework of Lipschitz continuous functions which are semiconcave outside the equilibrium. In the stochastic setting, Has'minskii [17,19] obtained a converse theorem for stability in probability of uncontrolled diffusion processes, strictly nondegenerate outside the equilibrium, by means of $\mathscr{C}^2$ Lyapunov functions, using the Maximum Principle and the properties of solutions of uniformly elliptic equation. Kushner proved in [20] a characterization of asymptotic uniform stochastic stability by means of continuous Lyapunov functions (here, however, the infinitesimal decrease condition is given in terms of the weak generator of the process). In [12] (see also [11]) the author extended the direct Lyapunov method by Has'minskii and Kushner to the study of stochastic open-loop stabilizability in probability in terms of merely semicontinuous Lyapunov functions which satisfy in the viscosity sense an appropriate infinitesimal decrease condition and provided also in this setting converse Lyapunov theorems.

The paper is organized as follows. In Section 2 we give the definition of stochastic open-loop a.s. stabilizability, in Section 3 we introduce the appropriate concept of Lyapunov function for the study of such stability. Section 4 is devoted to the viability properties of sublevel sets of Lyapunov functions. Section 5 contains the main results: the converse Lyapunov theorems. Finally, in Section 6 we give the extension to general equilibrium sets.