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A period-specific realization of linear continuous-time systems

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ABSTRACT

It is a well-known fact that a weighting pattern matrix has a periodic realization if and only if the matrix is separable and periodic. This fact, however, does not cope with a reasonable question of *period-specific realization problem*: for a given weighting pattern matrix and a given positive real number, find a periodic realization corresponding to the given matrix and with a period equal to the given number. This paper answers the question by showing that the period-specific realization problem has a solution if and only if the given weighting pattern matrix is separable and has a period equal to the given positive real number. To this end, two types of period-specific realizations are constructed. One is with a constant A-matrix whose dimension is not necessarily minimal among all possible period-specific realizations. The other is with a non-constant and periodically time-varying A-matrix whose dimension turned out minimal among all possible period-specific realizations.

1. Introduction

This paper is concerned with a classical realization problem, an inverse problem of retrieving a state-space representation from a given weighting pattern matrix which represents the input–output relation of a linear system, especially with a realization problem for linear *periodic* and *continuous-time* systems. This type of periodic realization problem was initiated by Silverman [1,2] and further developed in Shokoohi et al. [3], or discussed in somewhat different situation in Rahimi and Brockett [4]. Other types of realization problems have been investigated for linear periodic discrete-time systems [5–11], linear time-varying continuous-time systems [12–15], and linear time-invariant discrete-time systems [16–19]. A realization problem is one of the most fundamental problems in system theory; moreover, the realization problem for linear time-invariant discrete-time systems has been the first step toward practically important problems such as model reduction and system identification problems [19–21]. A linear periodic system is similar to a linear time-invariant system in the sense of the Floquet theory [22–26]. Therefore, the periodic realization problem has the potential to provide new insights into model reduction and system identification problems of linear periodic systems.

For the periodic realization problem for linear continuous-time systems, Silverman exhibited a necessary and sufficient condition for a weighting pattern matrix to be realized by a linear periodic system [1,2]. To be precise, it was proven for the necessity part that a weighting pattern matrix inherits a period from a given linear periodic system and for the sufficiency part that a linear periodic realization has *twice* the period of a given weighting pattern matrix. This suggests that the sufficiency part may accept an excessively wide class of linear periodic systems as candidates for periodic realization. This motivates us to initiate a reasonable question of *period-specific realization problem*: for a given weighting pattern matrix and a given positive real number, find a periodic realization corresponding to the given matrix and with a period equal to the given number. The first result of this paper gives an answer to the period-specific realization problem; it is shown that there is a period-specific realization as a solution to the problem if and only if the given weighting pattern matrix is separable and has a period equal to the given positive real number, where a linear periodic realization candidate with a dimensionally redundant constant A-matrix is employed.

The dimension of the proposed period-specific realization with constant A-matrix is twice the order of the given period-specifically realizable weighting pattern matrix; in contrast, the minimal dimension among all possible realizations is equal to the order of a given realizable weighting pattern matrix [12,13] and the minimal dimension among all possible periodic realizations is also equal to the order of a given periodically realizable weighting pattern matrix [1,2]. This also motivates us to investigate an open question of *period-specific minimal realization problem* to find the period-specific realization with the lowest dimension. The second result of this paper shows how the redundancy of dimension in our first result can be ultimately reduced by introducing a linear periodic realization candidate with a non-constant A-matrix.

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To this end, we firstly introduce the concept of *sign* corresponding to the specific period of the periodic weighting pattern matrices. Then, it is shown that, when the sign corresponding to the specific period is positive, the minimal dimension among all possible period-specific realizations is equal to the order of the weighting pattern matrix. It is also shown that, when the sign corresponding to the specified period is negative, the minimal dimension among possible period-specific realizations is larger exactly by one than the order of the weighting pattern matrix. The preliminary version of this paper was presented in [27].

We will use the following notation. The symbols $\mathbb{R}$, $\mathbb{R}^m$, and $\mathbb{R}^{m\times n}$ respectively denote the sets of all real numbers, real vectors with n-rows, and real matrices with n-rows and m-columns. $I_n \in \mathbb{R}^{n\times n}$ and $0_{m\times n} \in \mathbb{R}^{m\times n}$ respectively denote the identity matrix and the zero matrix (the subscripts will be omitted if their sizes are obvious). A^{T} denotes the transpose of a matrix A. $\det A$ and $\mathrm{tr}A$ respectively denote the determinant and trace of a square matrix A. $\mathrm{diag}[A_1, \dots, A_k]$ denote a block diagonal matrix in which the diagonal blocks are square matrices $A_1, \dots, A_k$ from the upper left and the off-diagonal blocks are all 0. C^k denotes the set of k-times continuously differentiable functions for $k = 1, \dots, \infty$. A function $f(t)$ is called T-periodic if $f(t) = f(t+T)$, $\forall t \in \mathbb{R}$.

2. Problem statement

Consider an n-dimensional linear T-periodic continuous-time control system with a state x, an input u, and an output y in the form

$$\dot{x} = A(t)x + B(t)u, \qquad y = C(t)x, \qquad \dot{x} := \frac{\mathrm{d}x}{\mathrm{d}t} \tag{1}$$

where $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$, $C(t) \in \mathbb{R}^{r\times n}$ are supposed to be T-periodic and continuous in t. Let C^1 real matrix-valued function $\Phi(t) \in \mathbb{R}^{n\times n}$ denotes the fundamental solution of $\dot{x} = A(t)x$ with the initial condition $\Phi(0) = I$. $\Phi(T)$ is called the *monodromy matrix* of $\dot{x} = A(t)x$ or that of $\Phi(t)$. Detailed discussions for linear periodic continuous-time systems can be found in the references [28–30].

The output under the initial condition $x(t_0) = 0$ is expressed by

$$y(t) = \int_{t_0}^{t} W(t,p)u(p)\,\mathrm{d}p,$$

where a continuous function $W(t,p) \in \mathbb{R}^{r\times m}$ is given by

$$W(t,p) := C(t)\Phi(t)\Phi(p)^{-1}B(p)$$

and is called a *weighting pattern matrix* of the system (1). Here, we prepare several terminologies for the weighting pattern matrix $W(t,p)$. $W(t,p)$ is called T-*periodic* if $W(t,p) = W(t+T, p+T)$, $\forall t, \forall p \in \mathbb{R}$, and is simply called *periodic* if it is T-periodic for a certain period $T > 0$. $W(t,p)$ is called *separable* if $W(t,p) = L(t)R(p)$, $\forall t, \forall p \in \mathbb{R}$ for some continuous real matrix-valued functions $L(t)$ and $R(t)$. The expression $W(t,p) = L_0(t)R_0(p)$, $\forall t, \forall p \in \mathbb{R}$ is called a *globally reduced form* if the columns and the rows of continuous real matrix-valued functions $L_0(t) \in \mathbb{R}^{r\times n_0}$ and $R_0(t) \in \mathbb{R}^{n_0\times m}$ are linearly independent over $\mathbb{R}$, respectively. Stated differently, this expression is globally reduced form if for every choice of nontrivial constant vectors $\xi, \eta \in \mathbb{R}^{n_0}$, $L_0(t)\xi \neq 0$ and $R_0(t)^{\mathrm{T}}\eta \neq 0$ on some sets of positive measure. The integer n_0 is uniquely defined independently of the choice of $L_0(t)$'s and $R_0(t)$'s in the globally reduced form and is called the *order* of $W(t,p)$.

A *realization problem* is an inverse problem of retrieving a system as (1) or equivalently a triplet of real matrix-valued functions $(A(t), B(t), C(t))$ for a given real weighting pattern matrix $W(t,p)$. In general, we can give a $W(t,p)$ which is not beforehand known to come from some linear system; in such a case we may have to answer to the realization problem by concluding "no solution". Since physical systems are usually approximated as linear systems with real coefficients, we will exclude any complex-valued weighting pattern matrices from the sources of the realization problem and any complex coefficient matrix from the candidates of solution to the realization problem, throughout this paper. Detailed discussions for linear time-varying continuous-time systems can be found in the references [12–15]. The following fact is most fundamental in realization theory.

Theorem 2.1. [12,13] *$W(t,p)$ has a realization if and only if it is separable.*

A *periodic realization problem* is a special type of realization problem: find a periodic realization in the form of a triplet of periodic real matrix-valued functions $(A(t), B(t), C(t))$ for a given real weighting pattern matrix $W(t,p)$. Let us recall a Silverman's result for this problem.

Theorem 2.2. [1,2] *$W(t,p)$ has a periodic realization if and only if it is separable and periodic.*

Notice that any specific period is not described in the statement of Theorem 2.2. As shown in the necessity part of the proof Theorem 2.2, it is immediate to obtain a stronger statement of "$W(t,p)$ has a T-periodic realization only if it is separable and T-periodic". In contrast, in the sufficiency part of the proof of Theorem 2.2, Silverman proved that "$W(t,p)$ has a $2T$-periodic realization if it is separable and T-periodic". This suggests that the sufficiency part may accept an excessively wide class of linear periodic systems as candidates for periodic realization with the specific period T.

This motivates us to initiate a reasonable question of *period-specific realization problem* to find a T-periodic realization in the form of a triplet of T-periodic real matrix-valued functions $(A(t), B(t), C(t))$ for a given pair of a real number $T > 0$ and a weighting pattern matrix $W(t,p)$.

A *minimal realization problem* is to find a realization with the lowest dimension for a given realizable weighting pattern matrix. The dimension of the minimal realization is equal to the order of a given realizable weighting pattern matrix [12,13]. A *periodic minimal realization problem* is to find a periodic realization with the lowest dimension for a given periodically realizable weighting pattern matrix. The dimension of the periodic minimal realization is also equal to the order of a given periodically realizable weighting pattern matrix [1,2]. In our period-specific realization problem, we restrict the candidate of realizations to have a specific period; therefore, there may be no solution to find a period-specific realization whose dimension is equal to the order of a given period-specifically realizable weighting pattern matrix.

This also motivates us to explore the further question of the *period-specific minimal realization problem* to find a T-periodic realization with the lowest dimension for a given pair of a real number $T > 0$ and a weighting pattern matrix $W(t,p)$.

3. Main results

3.1. Period-specific realization with constant A-matrix

In this section, we consider the bare period-specific realization problem. Firstly, we recall the realization procedure by Silverman in the sufficiency part of Theorem 2.2. This procedure is based on the following lemma.

Lemma 3.1. [1] *Let $W(t,p)$ be separable with the order $n_0 > 0$ and a globally reduced form $W(t,p) = L_0(t)R_0(p)$ be given. If $W(t,p)$ is T-periodic, then there exists a nonsingular real matrix $Q \in \mathbb{R}^{n_0\times n_0}$ such that*

$$L_0(t+T) = L_0(t)Q, \qquad R_0(t+T) = Q^{-1}R_0(t), \quad \forall t \in \mathbb{R}. \tag{2}$$