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Observer design for one-sided Lipschitz discrete-time systems

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ABSTRACT

This note focuses on state observer design for a general class of nonlinear discrete-time systems that satisfies the one-sided Lipschitz condition. It has been shown that this condition may encompass a large class of nonlinearities. However, challenging problems arise such as relevant choice of the Lyapunov function or non convexity of the obtained stability conditions. Both full-order and reduced-order observer designs are considered. In this work, the main contribution is to provide first some mathematical artifacts on the Lyapunov function to obtain simple and workable stability conditions, furthermore we show how to obtain LMI conditions to ensure asymptotic convergence. On the other hand, we extend the obtained results to $n-p$ reduced order observer design. High performances are shown through simulation results.

1. Introduction

Over the past two decades, there has been significant research activity on observer design for nonlinear systems; see [1–5] and the references inside. This topic was motivated by the fact that state estimation can be used for control design, diagnosis or, more recently, synchronization and unknown input recovery [6–8]. It is worth noticing, however, that most of the existing results concern continuous time systems with few extensions to discrete-time ones [9,10]. As no universal approach exists, state observers, in particular for nonlinear systems, are still a challenging and open problem. Beside the famous extended Kalman filter, we distinguish a simple and useful nonlinear state observer that is based on the solution of a Riccati-like equation and the Lipschitz condition, we refer the reader to the pioneering works in [11,12] and their extensions [13,14]. In recent contributions [13,14], limitations of this approach have been highlighted. Indeed, it has been shown that the solution of the Riccati-like equation depends strongly on the Lipschitz constant, i.e. more the latter is larger, the more difficult it is to find a solution to the Riccati-like equation.

In order to enlarge the domain of attraction and the class of nonlinear systems that can be considered, a useful and less general condition was recently introduced for observer design, that is the one sided Lipschitz condition. Interesting works on state observer design for this type of systems were recently developed in [15–20]; however the asymptotic stability condition leads to a challenging problem that is the resolution of bilinear matrix inequalities. More recently, Abbaszadeh and Marquez [20] have explored a more general Lipschitz condition with interesting mathematical artifacts to deduce LMI stability conditions. They show in particular inherent advantages with respect to the conservativeness induced by the classical Lipschitz condition. Inspired by their work we investigate here the problem of state observer design for one sided Lipschitz nonlinear discrete time systems. Indeed, it is worth noticing that the extension of the existing results on continuous time systems is a hard task and needs specific mathematical artifacts.

First, we provide a general formulation of a quadratic Lyapunov function and construct an extended state vector to formulate the asymptotic stability condition in Section 2. On the other hand, we provide simple and useful mathematical manipulations to deduce sufficient conditions for asymptotic convergence in terms of linear matrix inequalities. Furthermore, we extend the obtained results to $(n-p)$ reduced order observer design in Section 3. The latter may be interesting not only for real time applications but also may have less restrictive stability conditions. In the last section, relevant numerical examples are provided to show high performances of both techniques. Two illustrative examples are given in Section 4 to show the efficiency of the proposed approach.

Notations. In a matrix, the notation $(\star)$ is used for the blocks induced by symmetry. $\langle x, y\rangle = x^T y$ is the scalar product. $\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{x^T x}$ is the Euclidean vector norm. $|a|$ is the absolute

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value of the scalar a. $\lambda_i(A)$ is the ith eigenvalue of matrix A and $\|A\| = \sqrt{\lambda_{\max}(A^TA)}$ is the induced 2-norm of matrix A. If $A = A^T$, $\|A\| = \sqrt{\lambda_{\max}(A^TA)} = |\lambda_{\max}(A)|$. For a symmetric matrix A, $A > 0$ means that the matrix A is positive definite.

2. Full-order observer design

In this section, sufficient conditions for the existence of an observer are derived and a design procedure is presented. Let us consider the nonlinear system

$$\begin{cases} x(k+1) = Ax(k) + f(x(k), y(k)) \\ y(k) = Cx(k) \end{cases} \quad (1)$$

where $x(k) \in \mathbb{R}^n$ and $y(k) \in \mathbb{R}^p$ denote respectively the state and the linear output. A and C are constant matrices of adequate dimensions. $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is a real nonlinear vector field.

Our objective is to design an asymptotic observer from the measured output signals $y(k)$ to estimate the state $x(k)$. The following assumptions are made throughout this paper.

Assumption 1. 1. f is one-sided Lipschitz with respect to $x(k)$. i.e,

$$\langle f(x, y) - f(\hat{x}, y), x - \hat{x}\rangle \leqslant \rho\|x - \hat{x}\|^2, \quad \text{for any } x, \hat{x} \in \mathbb{R}^n,\ y \in \mathbb{R}^p \quad (2)$$

where ρ is the so-called one-sided Lipschitz constant which can be positive or negative.

2. f is quadratically inner-bounded with respect to $x(k)$. i.e,

$$\|f(x, y) - f(\hat{x}, y)\|^2 \leqslant \beta\|x - \hat{x}\|^2 + \gamma\langle x - \hat{x}, f(x, y) - f(\hat{x}, y)\rangle \quad (3)$$

where β and γ are real scalars.

Unlike the well-known Lipschitz condition, the constants ρ, β and γ can be positive, negative or zero. In addition, if the function f is Lipschitz, then it is also both one-sided Lipschitz and quadratically inner-bounded ($\beta > 0$ and γ), but the converse is not true (see [20]). The one-sided Lipschitz condition (2), considered in [15,18], provides a less conservative condition than the classical Lipschitz one. The concept of quadratic inner-boundedness (3), given in [20], is very useful to provide tractable LMI stability conditions.

The observer of system (1) is defined by the following form

$$\hat{x}(k+1) = A\hat{x}(k) + f(\hat{x}(k), y(k)) + K(y(k) - C\hat{x}(k)) \quad (4)$$

where $\hat{x}(k)$ denotes the estimate of the state vector $x(k)$ and K is the gain matrix to be computed.

Let $e(k) = x(k) - \hat{x}(k)$. Then from observer (4) and system (1), the dynamics of the state estimation error is described by

$$e(k+1) = (A - KC)e(k) + \Delta f_k \quad (5)$$

where $\Delta f_k = f(x(k), y(k)) - f(\hat{x}(k), y(k))$.

Using the above assumption, the following theorem provides sufficient conditions so that Eq. (4) is an asymptotic full-order observer for system (1).

Theorem 1. *Under Assumption 1, system (4) is an asymptotic observer for system (1) if there exist scalars $\alpha > 0$, $\mu_1 > 0$, $\mu_2 > 0$, ρ, β, γ and $\epsilon > 0$ and matrices $P = P^T > \alpha I_n$, $Q = Q^T > 0$, S and X that solve the following LMI*

$$\begin{bmatrix} P & S \\ S^T & Q \end{bmatrix} > 0 \quad (6)$$

and

$$\mathbb{N} < 0 \quad (7)$$

where $\mathbb{N}$ is given by

$$\mathbb{N} = \begin{bmatrix} \mathbb{N}_{11} & \mathbb{N}_{12} & 0 & \mathbb{N}_{14} & \mathbb{N}_{14} & 0 \\ \star & \mathbb{N}_{22} & \mathbb{N}_{23} & 0 & 0 & 0 \\ \star & \star & \mathbb{N}_{33} & 0 & 0 & \mathbb{N}_{23}^T \\ \star & \star & \star & -\eta^{-1}P & 0 & 0 \\ \star & \star & \star & \star & -\epsilon I_n & 0 \\ \star & \star & \star & \star & \star & -\epsilon^{-1}\alpha^2 I_n \end{bmatrix} \quad (8)$$

with

$$\begin{cases} \eta = 1 + 2(|\beta| + |\rho|) \\ \mathbb{S} = S - (\mu_1\gamma - \mu_2)I_n \\ \mathbb{N}_{11} = -P + 2(\mu_1\beta + \mu_2\rho)I_n \\ \mathbb{N}_{12} = \eta A^TP - \eta C^TX - \mathbb{S} \\ \mathbb{N}_{14} = A^TP - C^TX \\ \mathbb{N}_{22} = \eta P - Q - 2\mu_1 I_n \\ \mathbb{N}_{23} = S + \alpha(\gamma - 1)I_n \\ \mathbb{N}_{33} = Q - 2\alpha I_n. \end{cases} \quad (9)$$

Then, the gain for observer is given by $K = P^{-1}X^T$.

Proof. Let us consider the quadratic Lyapunov function

$$V_k = \begin{bmatrix} e(k) \\ \Delta f_k \end{bmatrix}^T \begin{bmatrix} P & S \\ S^T & Q \end{bmatrix} \begin{bmatrix} e(k) \\ \Delta f_k \end{bmatrix} \quad (10)$$

where Δf_k defined in (5) and $\begin{bmatrix} P & S \\ S^T & Q \end{bmatrix} > 0$. Moreover, the variation $\Delta V = V_{k+1} - V_k$ of this Lyapunov function is given by

$$\Delta V = e^T(k+1)Pe(k+1) - e^T(k)Pe(k) - \Delta f_k^T Q\Delta f_k + \Delta f_{k+1}^T Q\Delta f_{k+1} + 2e^T(k+1)S\Delta f_{k+1} - 2e^T(k)S\Delta f_k. \quad (11)$$

The one-sided Lipschitz and the quadratically inner-bounded conditions (2) and (3) give the following inequality

$$\begin{cases} \mu_2\rho e^T(k)e(k) - \mu_2 e^T(k)\Delta f_k \geqslant 0 \\ \mu_1\beta e^T(k)e(k) + \mu_1\gamma e^T(k)\Delta f_k - \mu_1\Delta f_k^T\Delta f_k \geqslant 0 \end{cases} \quad (12)$$

where μ_1 and μ_2 are arbitrary strictly positive scalars.

The following inequality is obtained by adding the left hand side of (12) to (11)

$$\begin{aligned} \Delta V \leqslant\ & e^T(k+1)Pe(k+1) + e^T(k)(-P + 2(\mu_2\rho + \mu_1\beta)I)e(k) \\ & + 2e^T(k+1)S\Delta f_{k+1} - 2e^T(k)(S + (\mu_2 - \mu_1\gamma)I)\Delta f_k \\ & - \Delta f_k^T(Q + 2\mu_1 I)\Delta f_k + \Delta f_{k+1}^T Q\Delta f_{k+1}. \end{aligned} \quad (13)$$

On the other hand, using the one-sided Lipschitz and the inner-bounded conditions (2) and (3) with the fact that $P > \alpha I_n$, it follows that

$$\begin{aligned} & |\rho|e^T(k+1)Pe(k+1) - \alpha e^T(k+1)\Delta f_{k+1} \\ & > \alpha|\rho|e^T(k+1)e(k+1) - \alpha e^T(k+1)\Delta f_{k+1} \geqslant 0, \end{aligned} \quad (14)$$

$$\begin{aligned} & |\beta|e^T(k+1)Pe(k+1) + \alpha\gamma e^T(k+1)\Delta f_{k+1} - \alpha\Delta f_{k+1}^T\Delta f_{k+1} \\ & > \alpha|\beta|e^T(k+1)e(k+1) + \alpha\gamma e^T(k+1)\Delta f_{k+1} \\ & \quad - \alpha\Delta f_{k+1}^T\Delta f_{k+1} \geqslant 0. \end{aligned} \quad (15)$$

Thus, by adding the left terms in inequalities (14) and (15) to (13), we get

$$\begin{aligned} \Delta V \leqslant\ & \eta e^T(k+1)Pe(k+1) + e^T(k) \\ & \times (-P + 2(\mu_2\rho + \mu_1\beta)I)e(k) \\ & - \Delta f_k^T(Q + 2\mu_1 I)\Delta f_k - 2e^T(k)(S + (\mu_2 - \mu_1\gamma)I)\Delta f_k \\ & + 2e^T(k+1)(S + \alpha(\gamma - 1)I)\Delta f_{k+1} \\ & + \Delta f_{k+1}^T(Q - 2\alpha I)\Delta f_{k+1}. \end{aligned} \quad (16)$$