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# Maximum principle for optimal control problems of forward–backward regime-switching system and applications ☆

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ABSTRACT

In this paper, we derive the stochastic maximum principle for optimal control problems of the forward–backward Markovian regime-switching system. The control system is described by forward–backward SDEs and modulated by continuous-time, finite-state Markov chains. We first obtain the necessary and sufficient conditions for the optimal control. Thereafter, we apply the maximum principle to recursive utility investment–consumption problems and LQ problems with Markovian regime-switching.



## 1. Introduction

The applications of a regime-switching model in finance and stochastic control have received significant attention in recent years. It performs better from the empirical point of view compared to the traditional system based on the diffusion processes. More specifically, it modulates the system with a continuous-time finite-state Markov chain with each state represents a regime of the system or a level of economic indicator. For example, in the stock market, the up-trend volatility of a stock tends to be smaller than its down-trend volatility (see Zhang [1]); therefore, it is reasonable to describe the market trends by a two-states Markov chain. The optimal control problems with jump Markov disturbances were first studied when the systems were described without Gaussian noise, see for example, in [2,3]. With the development of stochastic analysis and stochastic control theory, much work has been done on stability and stochastic control problems for the regime-switching system, such as [4–7]. The regime-switching model in economic and finance fields was first introduced by Hamilton in [8] to describe a time series model and then intensively investigated in the past two decades in mathematical finance. Based on the switching diffusion model, much work has been done in the fields of option pricing, portfolio management, Markowitz's mean–variance problem, risk management, etc. (see examples such as [9–12]).

Maximum principle was first formulated by Pontryagin in the 1950s and it converted the optimization problems into maximizing the corresponding Hamiltonian functions. Bismut [13] introduced the linear backward stochastic differential equations (BSDEs) as the adjoint equations, which was a milestone in the development of this theory. The general stochastic maximum principle was obtained by Peng in [14] by introducing the second order adjoint equations, which allowed the control enter in both the drift and diffusion coefficients while the control domain was nonconvex. Donnelly [15] investigated the sufficient maximum principle for the regime-switching model.

Pardoux and Peng proved the existence and uniqueness for the solution of the nonlinear BSDEs in [16], which has been extensively used in stochastic control and mathematical finance in the past two decades; see [17–19]. Duffie and Epstein [20] (independent of Pardoux and Peng) introduced the stochastic differential recursive utility by a kind of BSDEs which was a generalization of the standard additive utility. Generally, the recursive utility can be viewed as the solution of a BSDE. Peng first introduced the stochastic maximum principle for optimal control problems of the forward–backward control system when the control domain is convex in [21]. Dokuchaev and Zhou [22] studied a kind of maximum principle when the system dynamics were controlled BSDEs. Then, the forward–backward maximum principle was generalized and applied in finance; for example, see [23–27] and the references therein. In this paper, using the results about BSDEs with Markov chains in [28,29], we derive both the necessary and

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sufficient maximum principle for the forward–backward regime-switching model. To the authors' knowledge, it is the first time to investigate this system.

This paper is organized as follows. In Section 2, we give the preliminaries about BSDEs with Markov chains and the introductions to the optimal control problems. In Section 3, we derive the necessary maximum principle as well as sufficient optimality conditions. In Section 4, we give applications in the recursive utility investment–consumption problems and LQ problems with regime-switching.

## 2. Formulation of the optimal control problems

### 2.1. Preliminaries

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. $T > 0$ is a finite-time horizon. $\{B_t, 0 \le t \le T\}$ is a d-dimensional Brownian motion and $\{\alpha_t, 0 \le t \le T\}$ is a finite-state Markov chain with the state space given by $I = \{1, 2, \ldots, k\}$. The transition intensities are $\lambda(i, j)$ for $i \neq j$ with $\lambda(i, j)$ nonnegative and bounded. $\lambda(i, i) = -\sum_{j \in I \setminus \{i\}} \lambda(i, j)$. Let $\mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}$ be the filtration generated by $\{B_s, \alpha_s; 0 \le s \le T\}$ and augmented by all $\mathbb{P}$-null sets of $\mathscr{F}$.

Now, we recall some preliminary results about the integer random measure related to the Markov chain (see [28,29]); more details about the general integer random measure can be found in Jacod and Shiryaev [30].

Given an auxiliary measured space $(E, \mathscr{B}_E, \rho)$, where $\rho$ is a nonnegative $\sigma$-finite measure on $(E, \mathscr{B}_E)$. Let $\mu = (\mu(\mathrm{d}y, \mathrm{d}e))_{t\in[0,T], e\in E}$ be an integer-valued random measure on $([0, T] \times E, \mathscr{B}([0, T]) \otimes \mathscr{B}_E)$. Denote $\tilde{\mathscr{P}} = \mathscr{P} \otimes \mathscr{B}_E$, where $\mathscr{P}$ is the predictable sigma field on $\Omega \times [0, T]$. We assume that the compensator of $\mu$ is defined by $\zeta_t(\omega, e)\rho(\mathrm{d}e)\mathrm{d}t$ for a $\tilde{\mathscr{P}}$-measurable nonnegative uniformly bounded (random) function $\zeta$. Then, the compensatrix (compensated measure) is

$$\tilde{\mu}(\mathrm{d}t, \mathrm{d}e) = \mu(\mathrm{d}t, \mathrm{d}e) - \zeta_t(\omega, e)\rho(\mathrm{d}e)\mathrm{d}t.$$

In the following parts, by default, all equalities are assumed to be $\mathrm{d}\mathbb{P}$-a.s., $\mathrm{d}\mathbb{P} \otimes \mathrm{d}t$-a.e. or $\mathrm{d}\mathbb{P} \otimes \mathrm{d}t \otimes \zeta\,\mathrm{d}\rho$-a.e.

We also make the following notations.

- $|\cdot|$, the Euclidean norm in $\mathbb{R}^m$, the inner product is denoted by $\langle\cdot\rangle$.
- $\mathscr{M}_\rho$, the set of measurable functions from $(E, \mathscr{B}_E, \rho)$ to $\mathbb{R}$ endowed with the topology of convergence in measure.
- For $v \in \mathscr{M}_\rho$, define

$$|v|_t := \left[\int_E v(e)^2 \zeta_t(e)\rho(\mathrm{d}e)\right]^{1/2} \in \mathbb{R}_+ \cup \{+\infty\}. \tag{1}$$

- $\mathscr{L}^2_{\mathrm{d}}$, the space of square integrable $\mathbb{R}^{\mathrm{d}}$-valued $\mathscr{F}_T$-measurable random variable $\xi$, *s.t.*

  $\|\xi\| := (E\xi^2)^{1/2} < +\infty.$

- $S^p_{\mathrm{d}}$, the space of $\mathbb{R}^{\mathrm{d}}$-valued *càdlàg* process $Y$ *s.t.*

$$\|Y\|_{S^p_{\mathrm{d}}} := \left(E\left[\sup_{t\in[0,T]} |Y_t|^p\right]\right)^{1/p} < +\infty.$$

- $\mathscr{H}^2_{\mathrm{d}}$, the space of $\mathbb{R}^{1\times\mathrm{d}}$-valued predictable processes $Z$ *s.t.*

$$\|Z\|_{\mathscr{H}^2_{\mathrm{d}}} := \left(E\left[\int_0^T |Z_t|^2\,\mathrm{d}t\right]\right)^{1/2} < +\infty.$$

- $\mathscr{H}^2_\mu$, the space of $\tilde{P}$-measurable functions $V : \Omega \times [0, T] \times E \to \mathbb{R}$ *s.t.*

$$\begin{aligned}\|V\|_{\mathscr{H}^2_\mu} &:= \left(E\left[\int_0^T |V_t|_t^2\,\mathrm{d}t\right]\right)^{1/2} \\ &= \left(E\left[\int_0^T \int_E V_t(e)^2 \zeta_t(e)\rho(\mathrm{d}e)\mathrm{d}t\right]\right)^{1/2} \\ &< +\infty.\end{aligned} \tag{2}$$

**Remark 2.1.** It is noted that all the above spaces are Banach spaces.

Consider the following BSDEs,

$$\begin{cases}-\mathrm{d}Y_t = g(t, Y_t, Z_t, V_t)\mathrm{d}t - Z_t\mathrm{d}B_t - V_t(e)\widetilde{\mu}(\mathrm{d}s, \mathrm{d}e), \\ Y_T = \xi,\end{cases} \tag{3}$$

where $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^{1\times\mathrm{d}} \times \mathscr{M}_\rho \to \mathbb{R}$ and for any $y \in \mathbb{R}$, $z \in \mathbb{R}^{1\times\mathrm{d}}$, $v \in \mathscr{M}_\rho$, $g(t, y, z, v)$ is progressively measurable.

We make the following assumptions.

(H1) $\xi \in L^2$. (H2) $\|g(\cdot, 0, 0, 0)\|_{\mathscr{H}^2_1} < +\infty$.

(H3) $g$ is Lipschitz continuous with respect to $(y, z, v)$ in the sense that for any $t \in [0, T]$ and $(y, z, v), (y', z', v') \in \mathbb{R} \times \mathbb{R}^{1\times\mathrm{d}} \times \mathscr{M}_\rho$,

$$\begin{aligned}&|g(t, y, z, v) - g(t, y', z', v')| \\ &\quad \le C(|y - y'| + |z - z'| + |v - v'|_t).\end{aligned} \tag{4}$$

**Lemma 2.2** (*[28]*). *Under the assumptions* (H1)–(H3), *there exists a unique solution* $(Y_t, Z_t, V_t) \in (S^2_1 \times \mathscr{H}^2_{\mathrm{d}} \times \mathscr{H}^2_{\mathscr{V}})$ *for the BSDEs* (3).

Now we consider the random measure of Markov chains. The auxiliary measured space is $(I, \mathscr{B}_I, \rho)$, where $I$ is the state space of Markov chain $\alpha_t$, $\mathscr{B}_I$ is the sigma field of $I$, $\rho$ is defined by $\rho(\mathrm{d}j) = 1$ for $j \in I$.

Define $\mathscr{V}$ as the integer-valued random measure on $([0, T] \times I, \mathscr{B}([0, T] \otimes \mathscr{B}_I))$ which counts the jumps $\mathscr{V}_t(j)$ from $\alpha$ to state $j$ between time 0 and $t$. The compensator of $\mathscr{V}_t(j)$ is $1_{\{\alpha_t \neq j\}}\lambda(\alpha_t, j)\mathrm{d}t$, which means $\mathrm{d}\mathscr{V}_t(j) - 1_{\{\alpha_t \neq j\}}\lambda(\alpha_t, j)\mathrm{d}t := \mathrm{d}\tilde{\mathscr{V}}_t(j)$ is a martingale (compensated measure). Then canonical special semimartingale representation for $\alpha$ is given by

$$\mathrm{d}\alpha_t = \sum_{j\in I} \lambda(\alpha_t, j)(j - \alpha_t)\mathrm{d}t + \sum_{j\in I}(j - \alpha_{t-})\mathrm{d}\widetilde{\mathscr{V}}_t(j).$$

Denote $n_t(j) = 1_{\{\alpha_t \neq j\}}\lambda(\alpha_t, j)$. By a slight abuse of notation, we denote $\mathscr{M}_\rho = ((I, \mathscr{B}_I, \rho); \mathbb{R})$. Then the norms (1) and (2) of $\mathscr{M}_\rho$ and $\mathscr{H}^2_{\mathscr{V}}$ corresponding to the random measure of Markov chains are

$$|v|_t := \sum_{j\in I} \left[v(j)^2 n_t(j)\right]^{1/2} \in \mathbb{R}_+ \cup \{+\infty\}, \tag{5}$$

$$\|V\|_{\mathscr{H}^2_{\mathscr{V}}} := \left(E\left[\int_0^T \sum_{j\in I} V_t(j)^2 n_t(j)\mathrm{d}t\right]\right)^{1/2} < +\infty, \tag{6}$$

and the BSDEs with Markov chains are of the form:

$$\begin{cases}-\mathrm{d}Y_t = g(t, Y_t, Z_t, V_t)\mathrm{d}t - Z_t\mathrm{d}B_t - \displaystyle\sum_{j\in I} V_t(j)\mathrm{d}\tilde{\mathscr{V}}_t(j), \\ Y_T = \xi.\end{cases} \tag{7}$$

### 2.2. Formulation of the problems

We will consider optimal control problems of the following stochastic control system, which is a coupled forward–backward