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# Forward–backward linear quadratic stochastic optimal control problem with delay

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ABSTRACT

This paper is concerned with one kind of forward–backward linear quadratic stochastic control problem whose system is described by a linear anticipated forward–backward stochastic differential delayed equation. The explicit form of the optimal control is derived. Optimal state feedback regulators are studied in two special cases. For the case with delay in just the control variable, the optimal state feedback regulator is obtained by the Riccati equation. For the other case with delay in just the state variable, the optimal state feedback regulator is analyzed by the value function approach.



## 1. Introduction

It is well known that the *linear quadratic* (LQ) stochastic control problem is an important yet fascinating class of stochastic control ones, and the theoretical results of this problem have lots of significant impacts on a wide range of engineering, managing and financial applications (see [1]).

Since the work of Pardoux–Peng [2], the theory of *backward stochastic differential equations* (BSDEs) has been studied systematically. Research on related fields and their applications has become a notable endeavor among researchers in mathematical finance, optimal control, stochastic games and partial differential equations. Stochastic control problems, whose systems are driven by BSDEs or *forward–backward stochastic differential equations* (FBSDEs) have been widely studied by many authors, see [3–9]. Backward and forward–backward LQ stochastic control problems can be seen in [10–13] and in [14,8,9], respectively.

This sort of problem with delay emerges naturally when we study various natural and social phenomena under uncertainty, which are a class of *time-inconsistent* control problems. In contrast with standard stochastic control problems, the cost functional and underlying systems involve their present value as well as their previous information, i.e., their behavior at time $t$ not only depends on the current situation but also on a finite part of their past history. Such as the evolution of the stock price and other stochastic dynamical systems. Such models may be formulated as *stochastic differential delayed equations* (SDDEs) (or SDEs with finite aftereffects), which are a natural generalization of the classical SDEs. Stochastic control problems for systems with delay have attracted many researchers' attention since the initial work of Kolmanovsky–Maizenberg [15], where a linear delayed system with a quadratic cost functional was considered. This kind of stochastic control problem now appears widely in engineering, finance and other research fields (see [16–18] and the references therein).

LQ stochastic control problems with delay were studied in [15,16,19], etc. Recently, [20] obtained a maximum principle for one kind of stochastic control problem with delay. In their paper, they encountered some new type of linear BSDEs when introducing the adjoint equation. This new type of BSDEs was already introduced by Peng–Yang [21] for the general nonlinear case and they called them *anticipated BSDEs* (ABSDEs). Moreover, a duality relation between SDDE and ABSDE has also been explored. To the best of our knowledge, backward or forward–backward LQ stochastic control problems with delay have never been studied in the literature. In this paper, we mainly consider one kind of forward–backward LQ stochastic control problem with delay. Our work distinguishes itself from the recent work of [19] in the following aspects:

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On one hand, some interesting yet important problems in mathematical economics and mathematical finance, especially in the optimization problem, can be formulated as FBSDEs. For example, the interesting optimal portfolio choice for a "large" investor which is considered in [22] can be considered as applications of FBSDEs. However, the control system of the practical background often involves delayed information, which can be naturally formulated as a linear *anticipated forward–backward stochastic differential delayed equation* (AFBSDDE). This motivates us to study this kind of problem with stochastic delayed systems. Also, since the time delayed model is one of the time-consistent ones, we believe that research on AFBSDDEs and their wide applications in mathematical finance is an important yet fascinating topic. In general, the study of such special time-inconsistent control problems including time delay is still in its infancy.

On the other hand, our control system is more general than that in [19]. Moreover, a state feedback optimal control regulator is obtained in some special cases while [19] did not. For the case with delay in just the control variable, we derive the optimal state feedback regulator by the Riccati equation. For the case with delay in just the state variable, we obtain the optimal state feedback regulator by the value function approach.

The rest of this paper is organized as follows. In Section 2, we state some preliminary results about SDDEs coupled with anticipated BSDEs. In Section 3, we study the forward–backward LQ stochastic control problem with delay and give our main result. Two special cases are investigated in Section 4 and optimal state feedback regulators are obtained by different methods. Finally in Section 5, we present some concluding remarks.

## 2. Preliminary result of SDDE coupled with anticipated BSDE

*Throughout this paper,* $\mathbf{R}^n$ *denotes the n-dimensional Euclidean space.* $\langle\cdot,\cdot\rangle$ *and* $|\cdot|$ *denote the scalar product and norm in the Euclidean space, respectively.* $\top$ *in the superscripts denotes the transpose of a matrix.*

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ be a stochastic basis satisfying the usual conditions. We suppose that the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by a standard $d$-dimensional Brownian motion $\{W(t)\}_{t\geq 0}$. Let $T > 0$ be the finite time duration and $\delta > 0$ be the constant time delay. We denote by $C([-\delta, 0]; \mathbf{R}^n)$ the space of continuous functions $\varphi(t) : [-\delta, 0] \to \mathbf{R}^n$ satisfying $\sup_{-\delta\leq t\leq 0}|\varphi(t)| < \infty$, by $L^2(\mathcal{F}_t; \mathbf{R}^n)$ the space of $\mathbf{R}^n$-valued $\mathcal{F}_t$-measurable random variables $\zeta$ satisfying $\mathbb{E}|\zeta|^2 < \infty, \forall t \in [0, T]$, and by $L^2_{\mathcal{F}}([0, T]; \mathbf{R}^n)$ the space of $\mathbf{R}^n$-valued $\mathcal{F}_t$-adapted processes $\psi_t$ satisfying $\mathbb{E}\int_0^T |\psi_t|^2 dt < \infty$. $\mathbb{E}^{\mathcal{F}_t}[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_t]$ denotes the conditional expectation under filtration $\mathcal{F}_t$.

We first consider the following coupled AFBSDDE:

$$\begin{cases} dx(t) = b(t, x(t), y(t), z(t), x(t-\delta))dt \\ \quad +\sigma(t, x(t), y(t), z(t), x(t-\delta))dW(t), \\ -dy(t) = f(t, x(t), y(t), z(t), \mathbb{E}^{\mathcal{F}_t}[y(t+\delta)], \\ \quad \mathbb{E}^{\mathcal{F}_t}[z(t+\delta)])dt - z(t)dW(t), \quad t \in [0, T], \\ x(t) = \xi(t), \quad t \in [-\delta, 0], \\ y(T) = g(x(T)), \qquad y(t) = \varphi(t), \quad t \in (T, T+\delta], \\ z(t) = \phi(t), \quad t \in [T, T+\delta]. \end{cases} \tag{2.1}$$

In the above, for all $t \in [0, T]$, $b : \Omega \times [0, T] \times \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^{m\times d} \times L^2(\mathcal{F}_s; \mathbf{R}^n) \to L^2(\mathcal{F}_t; \mathbf{R}^n)$, $\sigma : \Omega \times [0, T] \times \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^{m\times d} \times L^2(\mathcal{F}_s; \mathbf{R}^n) \to L^2(\mathcal{F}_t; \mathbf{R}^n)$, $f : \Omega \times [0, T] \times \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^{m\times d} \times L^2(\mathcal{F}_r; \mathbf{R}^n) \times L^2(\mathcal{F}_{r'}; \mathbf{R}^n) \to L^2(\mathcal{F}_t; \mathbf{R}^n)$, $g : \Omega \times \mathbf{R}^n \to \mathbf{R}^m$ are given, where $s \in [-\delta, t]$, $r, r' \in [t, T+\delta]$. $\xi(\cdot) \in C([-\delta, 0]; \mathbf{R}^n)$, $\varphi(\cdot) \in L^2_{\mathcal{F}}([T, T+\delta]; \mathbf{R}^m)$, $\phi(\cdot) \in L^2_{\mathcal{F}}([T, T+\delta]; \mathbf{R}^{m\times d})$.

We denote $x(t-\delta), \mathbb{E}^{\mathcal{F}_t}[y(t+\delta)], \mathbb{E}^{\mathcal{F}_t}[z(t+\delta)]$ by $x(\delta), y(+\delta), z(+\delta)$, respectively. Given an $m \times n$ full-rank matrix $G$, we use the following notations:

$$\Gamma := \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

$$\mathcal{A}(t, \Gamma, x(\delta), y(+\delta), z(+\delta)) := \begin{pmatrix} -G^\top f(t, \Gamma, y(+\delta), z(+\delta)) \\ Gb(t, \Gamma, x(\delta)) \\ G\sigma(t, \Gamma, x(\delta)) \end{pmatrix},$$

here $G\sigma \equiv (G\sigma_1, G\sigma_2, \ldots, G\sigma_d)$.

**Definition 2.1.** A triple of processes $(x(\cdot), y(\cdot), z(\cdot)) : \Omega \times [-\delta, T] \times [0, T+\delta] \times [0, T+\delta] \to \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^{m\times d}$ is called an adapted solution to (2.1) if $(x(\cdot), y(\cdot), z(\cdot)) \in L^2_{\mathcal{F}}([-\delta, T]; \mathbf{R}^n) \times L^2_{\mathcal{F}}([0, T+\delta]; \mathbf{R}^m) \times L^2_{\mathcal{F}}([0, T+\delta]; \mathbf{R}^{m\times d})$ satisfies the coupled AFBSDDE (2.1).

We assume that the following hypothesis holds.

$$\begin{cases} \text{(i) For each } \Gamma, \mathcal{A}(t, \Gamma, x(\delta), y(+\delta), z(+\delta)) \in L^2_{\mathcal{F}}([0, T]); \\ \text{(ii) For each } x \in \mathbf{R}^n, g(x) \in L^2(\mathcal{F}_T; \mathbf{R}^m) \\ \quad \text{is Lipschitz in } x; \\ \text{(iii) There exists a constant } C > 0 \\ \quad \text{such that for all } t \in [0, T], \\ |\mathcal{A}(t, \Gamma, x(\delta), y(+\delta), z(+\delta)) - \mathcal{A}(t, \bar{\Gamma}, \bar{x}(\delta), \bar{y}(+\delta), \bar{z}(+\delta))| \\ \quad \leq C(|\hat{\Gamma}| + |\hat{x}(\delta)| + \mathbb{E}^{\mathcal{F}_t}[|\hat{y}(+\delta)| + |\hat{z}(+\delta)|]), \end{cases} \tag{H2.1}$$

$$\begin{cases} \displaystyle\int_0^T \langle \mathcal{A}(t, \Gamma, x(\delta), y(+\delta), z(+\delta)) \\ \quad -\mathcal{A}(t, \bar{\Gamma}, \bar{x}(\delta), \bar{y}(+\delta), \bar{z}(+\delta)), \Gamma - \bar{\Gamma}\rangle dt \\ \displaystyle\leq \int_0^T [-\beta_1|G\hat{x}|^2 - \beta_2(|G^\top\hat{y}|^2 + |G^\top\hat{z}|^2)]dt, \\ \langle g(x) - g(\bar{x}), G(x-\bar{x})\rangle \geq \mu_1|G\hat{x}|^2, \end{cases} \tag{H2.2}$$

for all $\Gamma = (x, y, z)$, $\bar{\Gamma} = (\bar{x}, \bar{y}, \bar{z})$, $\hat{x} = x - \bar{x}$, $\hat{y} = y - \bar{y}$, $\hat{z} = z - \bar{z}$. Here $\beta_1$, $\beta_2$ and $\mu_1$ are given nonnegative constants with $\beta_1 + \beta_2 > 0$, $\beta_2 + \mu_1 > 0$. Moreover we have $\beta_1 > 0$, $\mu_1 > 0$ (resp., $\beta_2 > 0$), when $m > n$ (resp., $m < n$).

The following result in [19] is the existence and uniqueness result.

**Lemma 2.1.** *Let (H2.1) and (H2.2) hold. Then there exists a unique adapted solution* $(x(\cdot), y(\cdot), z(\cdot))$ *to the coupled AFBSDDE (2.1).*

*Moreover, we consider the following general coupled AFBSDDE:*

$$\begin{cases} dx(t) = b(t, x(t), By(t), Dz(t), x(t-\delta))dt \\ \quad +\sigma(t, x(t), By(t), Dz(t), x(t-\delta))dW(t), \\ -dy(t) = f(t, x(t), y(t), z(t), \mathbb{E}^{\mathcal{F}_t}[y(t+\delta)], \mathbb{E}^{\mathcal{F}_t}[z(t+\delta)])dt \\ \quad -z(t)dW(t), \quad t \in [0, T], \\ x(t) = \xi(t), \quad t \in [-\delta, 0], \\ y(T) = g(x(T)), \qquad y(t) = \varphi(t), \quad t \in (T, T+\delta], \\ z(t) = \phi(t), \quad t \in [T, T+\delta], \end{cases} \tag{2.2}$$

*where* $D = (D_1, \ldots, D_d)$, $B, D_1, \ldots, D_d$ *are* $k \times m$ *matrices.* $(x(\cdot), y(\cdot), z(\cdot)) \in \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^{m\times d}$, $b, \sigma, f, g$ *have appropriate dimensions. We denote*

$$\mathcal{A}(t, \Gamma, x(\delta), y(+\delta), z(+\delta)) := \begin{pmatrix} -G^\top f(t, \Gamma, y(+\delta), z(+\delta)) \\ Gb(t, x, By, Dz, x(\delta)) \\ G\sigma(t, x, By, Dz, x(\delta)) \end{pmatrix},$$