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Optimal semistable control for continuous-time linear systems

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ABSTRACT

In this paper, we develop a new $\mathcal{H}_2$ semistability theory for linear dynamical systems. Specifically, necessary and sufficient conditions based on the new notion of weak semiobservability for the existence of solutions to the semistable Lyapunov equation are derived. Unlike the standard $\mathcal{H}_2$ optimal control problem, a complicating feature of the $\mathcal{H}_2$ optimal semistable control problem is that the semistable Lyapunov equation can admit multiple solutions. We characterize all the solutions using matrix analysis tools. With this theory, we present a new framework to design $\mathcal{H}_2$ optimal semistable controllers for linear coupled systems by converting the original optimal control problem into a convex optimization problem.

1. Introduction

Multiagent networked systems cover a very broad spectrum of applications including cooperative control of unmanned air vehicles, autonomous underwater vehicles, distributed sensor networks, air and ground transportation systems, swarms of air and space vehicle formations, and congestion control in communication networks, to cite but a few examples. A unique feature of the closed-loop dynamics under any control algorithm in multiagent networks is the existence of a continuum of equilibria representing a desired state of convergence. For example, the consensus problem [1–3] requires all the states in the multiagent network achieve the same value eventually. But this value is not determined *a priori*. Under such dynamics, the desired limiting state is not determined completely by the system dynamics, but depends on the initial system state as well [2,3]. In such systems possessing a continuum of equilibria, *semistability*, and not asymptotic stability, is the relevant notion of stability [2,3]. Semistability is the property whereby every trajectory that starts in a neighborhood of a Lyapunov stable equilibrium converges to a (possibly different) Lyapunov stable equilibrium. Semistability then implies Lyapunov stability, and is implied by asymptotic stability. Thus, to design a control algorithm to achieve cooperative tasks like consensus for multiagent systems, one has to incorporate the notion of semistability into the controller design so that a stable multiagent system is guaranteed [4].

Semistability was first introduced in [5] for linear systems, and applied to matrix second-order systems in [6]. Nonlinear extensions were considered in [7,8], which give several stability results for systems having a continuum of equilibria based on nontangency and arc length of trajectories, respectively. Refs. [2,3] build on the results of [7,8] and give semistable stabilization results for nonlinear network dynamical systems. Optimal semistable stabilization, however, has never been considered in the literature.

One may argue that for a linear system, the case of continuum of equilibria can only arise when the system matrix is singular. But this is not the generic case, because any perturbations would make a singular matrix to be nonsingular. That is, given a constant linear system in real case, the probability for it to be a type singular is zero. So, it is of less practical interest to consider such a very special case. We argue that this statement is not true for a large class of linear systems which come from multiagent consensus problems. We detail our presentation in Section 2 to show that this special case is indeed practically important and the controller design associated with such a case is worth a thorough investigation due to its complex nature shown in three examples in Section 3.

In this paper, we develop a new $\mathcal{H}_2$ optimal semistable control framework for linear dynamical systems. Specifically, necessary and sufficient conditions based on the new notion of weak semiobservability for the existence of solutions to the semistable Lyapunov equation are derived. Unlike the standard $\mathcal{H}_2$ optimal control problem, a complicating feature of the $\mathcal{H}_2$ optimal semistable control problem is that the semistable Lyapunov equation can admit multiple solutions. We characterize all the solutions using matrix analysis tools. With this theory, we present a new framework to design $\mathcal{H}_2$ optimal semistable controllers for linear coupled systems by converting the original optimal control problem into a convex optimization problem. It is important to note that the proposed semistable control framework is different

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from the traditional servo control mechanism since we do not use the error signal to serve as a feedback signal in the closed-loop system. Furthermore, the results of this paper are the initial step towards analysis and synthesis of linear systems having a continuum of equilibria, which could be viewed as an alternative, but not more general method to those approaches in [9,10].

The rest of the paper is organized as follows. Section 3 gives the formulation of the optimal semistable control problem and its well-posedness problem by illustrating three interesting examples. It follows from these three examples that the complexity of the well-posedness problem for optimal semistable control is far beyond the current $\mathcal{H}_2$ control theory. Section 4 first explores necessary conditions for optimality and semistability of linear systems. Based on these results, necessary and sufficient conditions for semistability of linear systems are then developed by introducing the notion of weak semiobservability in Section 4. These necessary and sufficient conditions turn out to be the bridge establishing the equivalent optimal semistable control problems. The new convex optimization equivalent formulation is shown in Section 5 in which an equivalent nonconvex optimization problem and an equivalent convex optimization problem are proposed. Finally, some concluding remarks are provided in Section 6.

2. Perturbation to a linear system with a continuum of equilibria: the drifting phenomenon

In this section, we use a linear consensus protocol with imperfect information as an example to show that linear systems with singular matrices are indeed widespread in multiagent coordination and perturbation to these systems has a serious effect on stability. Specifically, we will illustrate one particular serious consequence of imperfect information that can cause agent-based networks to become unbounded even when the corresponding network with perfect communication is bounded. This study was initiated in [11].

Consider a networked multiagent system consisting of n agents, whose dynamics are $\dot{x}_i = u_i$, where $x_i \in \mathbb{R}$ is the state and u_i is the control input. The agents can communicate with each other according to a graph G (also called a *topology* of the network) such that two agents can exchange information only if there is an edge between them. Denote by $\mathcal{N}_i$ the *neighborhood* of agent i and by $[a_{ij}]$ the adjacency matrix of G. It is well known (see, e.g., [1]) that if the graph G is strongly connected and balanced, then under the following distributed control law (also called a *protocol* in this context):

$$u_i = \sum_{j\in\mathcal{N}_i} a_{ij}(x_j - x_i), \tag{1}$$

all the agents will asymptotically reach the same value such that $x_i(t) \to \bar{x}$ as $t \to \infty$ for some constant $\bar{x}$ (which depends on the initial states of the agents). If that is the case, the protocol (1) is also called a *consensus protocol*. When the consensus a is the average of the states, $a = \frac{1}{n}\sum_{i=1}^{n} x_i$, the protocol is called the *average consensus protocol*. There are also other variations [12,13] of the average consensus protocol (1); it is referred to the paper [14] for a survey of consensus algorithms and their applications.

An implicit assumption in the protocol (1), as in almost all of the reported literature (e.g., [1,12,13,15,16]), is that the information agent i received from agent j is perfectly x_j. In this research, we consider the situation where the information is not perfect such that what agent i received from agent j is indeed $x_j + w_j$, where w_j is the *uncertainty* (agent i can still have the exact x_i because that information is readily available to agent i). The source of uncertainty could come from physical communication channels between the agents (such as noise of communication channels) but it can also be a quantization error resulting from converting real numbers into finite-bit data (for storing and digital communication). For the sake of presentation, we assume that the uncertainty w_i is the same for all recipients of x_i but in general, we can have different w_{ij} for different communication links. In the presence of the uncertainty w_i, the Eq. (1) becomes

$$u_i = \sum_{j\in\mathcal{N}_i}(x_j + w_j - x_i).$$

The collective dynamics of the network is

$$\dot{x} = -Lx - Aw \tag{2}$$

where $x = [x_1, \ldots, x_n]^{\mathrm{T}}$ is the state vector, $w = [w_1, \ldots, w_n]^{\mathrm{T}}$ is the uncertainty, and L is the Laplacian and A is the adjacency matrix of G. Because L has zero eigenvalue, the system with $w = 0$ is only marginally stable, and in the presence of a bounded w, the state x could go to infinity. Another way to see this drifting phenomenon is to consider the dynamics of $\alpha = \sum_{i=1}^{n} x_i = \mathbf{1}^{\mathrm{T}}x$, where $\mathbf{1}$ is the vector whose elements are all 1. The dynamics of α are

$$\dot{\alpha} = \mathbf{1}^{\mathrm{T}}(-Lx - Aw) = -\mathbf{1}^{\mathrm{T}}Aw \tag{3}$$

since $\mathbf{1}^{\mathrm{T}}L = 0$. Therefore, if $\int_0^t \mathbf{1}^{\mathrm{T}}Aw(s)\mathrm{d}s \to \infty$ as $t \to \infty$, then $\alpha(t) \to \infty$ as $t \to \infty$. In particular, if $\mathbf{1}^{\mathrm{T}}Aw$ is a nonzero constant (one can always pick such a nonzero w since $\mathbf{1}^{\mathrm{T}}Aw$ is continuous in w), no matter how small w is, it will grow unbounded and thus, $|x| \to \infty$.

Thus, one has to be careful when designing control laws for linear systems with a continuum of equilibria and it is absolutely necessary to discuss optimal control of such systems.

3. Problem formulation

The notation we use in this paper is fairly standard. Specifically, $\mathbb{R}$ (resp., $\mathbb{C}$) denotes the set of real (resp., complex) numbers, $\mathbb{R}^n$ (resp., $\mathbb{C}^n$) denotes the set of $n \times 1$ real (resp., complex) column vectors, $\mathbb{R}^{n\times m}$ (resp., $\mathbb{C}^{n\times m}$) denotes the set of $n \times m$ real (resp., complex) matrices, $(\cdot)^{\mathrm{T}}$ denotes transpose, $(\cdot)^*$ denotes complex conjugate transpose, $(\cdot)^\#$ denotes the group generalized inverse, and I_n or I denotes the $n \times n$ identity matrix. Furthermore, we write $\|\cdot\|$ for the Euclidean vector norm, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ for the range space and the null space of a matrix A, $\mathrm{rank}(A)$ for the rank of a matrix A, $\mathrm{spec}(A)$ for the spectrum of the square matrix A, $\mathrm{tr}(\cdot)$ for the trace operator, and $A \geq 0$ (resp., $A > 0$) to denote the fact that the Hermitian matrix A is positive semidefinite (resp., definite). Finally, we write $\mathcal{B}_\varepsilon(x)$, $x \in \mathbb{R}^n$, $\varepsilon > 0$, for the open ball with *radius* ε and *center* x, $\otimes$ for the Kronecker product, $\oplus$ for the Kronecker sum, and $\mathrm{vec}(\cdot)$ for the column stacking operator.

In this paper, we consider q continuous-time linear systems $\mathcal{G}_i$, $i = 1, 2, \ldots, q$, given by

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t), \qquad x_i(0) = x_{i0}, \quad t \geq 0, \tag{4}$$

where $x_i(t) \in \mathbb{R}^n$ denotes states of the ith system, $u_i(t) \in \mathbb{R}^m$ denotes inputs of the ith system, $A_i \in \mathbb{R}^{n\times n}$, and $B_i \in \mathbb{R}^{n\times m}$. Note that this model includes both *homogeneous systems* [17] in which all $\mathcal{G}_i$s are identical and *heterogeneous systems* [18] in which $\mathcal{G}_i$s are different. We can rewrite the overall system as the compact form

$$\dot{x} = Ax + Bu, \tag{5}$$

where $x \triangleq [x_1^{\mathrm{T}}, \ldots, x_q^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{nq}$, $u = [u_1^{\mathrm{T}}, \ldots, u_q^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{mq}$, $A \triangleq$ block-diag $[A_1, \ldots, A_q]$, and $B \triangleq$ block-diag$[B_1, \ldots, B_q]$.

In this section, we consider the linear controller design $u_i = K_i x$ so that the closed-loop system is *semistable*, that is, $\lim_{t\to\infty} x_i(t) = \alpha_i$, $i, = 1, 2, \ldots, n$, where α_i is the final value determined by the initial condition, and the cost function

$$J(u, x_0) = \int_0^\infty [(x(s) - \alpha)^{\mathrm{T}}Q(x(s) - \alpha) + (u(s) - \beta)^{\mathrm{T}}R(u(s) - \beta)]\mathrm{d}s \tag{6}$$