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Explicit solution of min–max MPC with additive uncertainties and quadratic criterion

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Abstract

Min–max model predictive control (MMMPC) is one of the strategies proposed to control plants subject to bounded uncertainties. This technique is very difficult to implement in real time because of the computation time required. Recently, the piecewise affine nature of this control law has been proved for unconstrained linear systems with quadratic performance criterion. However, no algorithm to compute the explicit form of the control law was given. This paper shows how to obtain this explicit form by means of a constructive algorithm. An approximation to MMMPC in the presence of constraints is presented based on this algorithm.

Keywords: Predictive control; Robust control; Uncertain systems

1. Introduction

Model predictive control (MPC) is one of the few control techniques able to cope with model uncertainties in an explicit way [7]. One approach used in MPC when uncertainties are present, is to minimize the objective function for the worst possible case. This strategy is known as min–max model predictive control (MMMPC) and was originally proposed in [27] in the context of robust receding horizon control and in [8] in the context of robust MPC. All MMMPC techniques for constrained and unconstrained linear uncertain systems have a great computational burden in common (see [14,21,25]) which limits the range of processes to which they can be applied. Few applications can be found in literature even for slow dynamics or complex simulated models (see [11,18]). In order to overcome the computational burden, several works have been proposed in the literature (see for example [12,13,23]). Even though, the implementation of robust MPC on real systems remain an open question.

It was shown in [5] that constrained MPC could be solved using multiparametric linear or quadratic programming (depending on the objective function). In this way an easily implemented explicit solution can be obtained. These types of results were extended to min–max controllers for linear uncertain systems with l_1 or l_∞ norms in [4,10]. The piecewise affine nature for quadratic cost functions has also been proved by other means in [19,20]. However, these works do not include an algorithm to obtain the explicit form of the control law.

This paper presents an algorithm that computes the explicit form of an unconstrained MMMPC controller with a quadratic cost function. The range of processes to which, in practice, these controllers can be applied is thus considerably broadened. Moreover, the constrained formulation is taken into account in the paper. An approximated min–max controller based on the explicit solution of the unconstrained formulation is presented. This controller minimizes an upper bound of the cost function and the optimization problem to solve is a quadratic programming problem.

The paper is organized as follows: Section 2 introduces the controller and its related optimization problem. Some properties of the min–max problem are shown in Section 3. The characterization of the regions in which the state space

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can be partitioned is presented in Section 4. In Section 5 the algorithm for exploring the state space and computing the explicit controller is presented. In Section 6 constraint handling is addressed. Section 7 illustrates the results presented in the paper by means of some simulated examples. Finally, we present concluding remarks in Section 8.

2. Min–max MPC with additive bounded uncertainties

Consider the discrete invariant time linear system with bounded uncertainties

$$x_{k+1} = Ax_k + Bu_k + Dw_k, \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$ is the state, $u_k \in \mathbb{R}^{n_u}$ is the control input, and $w_k \in \mathbb{R}^{n_w}$ is the uncertainty. The uncertainty is supposed to be bounded; i.e. $\|w_k\|_\infty \leqslant \varepsilon$.

Open loop min–max MPC obtains a single control input sequence that minimizes the worst case cost (see [8,17,26]) in which the predictions are computed in an open-loop manner (although the resulting controller is a feedback controller). These controllers are based on the solution of a single min–max problem optimizing a single control sequence for all possible values of the uncertainty. This formulation is known to be conservative because it underestimates the set of feasible input trajectories [21]. One solution proposed in the literature is to minimize a sequence of control corrections efforts to a given linear feedback stabilizing control law for the nominal plant. In this way, some kind of feedback is introduced in the prediction without increasing the computational effort (see [3,15]). The control input is given by $u_k = Kx_k + v_k$, where K is chosen in order to achieve some desired property for the non-constrained problem such as stability or LQR optimality. The MPC controller will compute the optimal sequence of correction control inputs v_k. The dynamics of the system can be rewritten as

$$x_{k+1} = A_K x_k + Bv_k + Dw_k,$$

where $A_K = (A + BK)$.

Consider a sequence $\mathbf{v} = \{v_0, v_1, \ldots, v_{N-1}\}$ of correction control inputs and $\mathbf{w} = \{w_0, w_1, \ldots, w_{N-1}\}$ a possible sequence of input disturbances to the system over a prediction horizon N. The objective function is defined as a quadratic performance index of the form

$$J(\mathbf{v}, \mathbf{w}, x) = \sum_{j=0}^{N-1} [x_j^{\mathrm{T}} Q x_j + u_j^{\mathrm{T}} R u_j] + x_N^{\mathrm{T}} P x_N,$$

where x_j and u_j are the predicted state and input of time j taking into account the uncertainty $\mathbf{w}$. The initial state is $x_0 = x$. Weighting matrices $Q = Q^{\mathrm{T}} \geqslant 0$ and $P = P^{\mathrm{T}} \geqslant 0$ are positive semi-definite, and $R = R^{\mathrm{T}} > 0$ is positive definite.

Taking into account (1), variables x_j and u_j are given by linear functions of x, $\mathbf{v}$ and $\mathbf{w}$, namely

$$x_j = A_K^j x + \sum_{i=1}^{j} A_K^{i-1} B v_{j-i} + \sum_{i=1}^{j} A_K^{i-1}\, D w_{j-i},$$
$$u_j = Kx_j + v_j. \tag{2}$$

Min–max MPC [8] is based on finding the control sequence $\mathbf{v}$ that minimizes $J(\mathbf{v}, \mathbf{w}, x)$ for the worst possible case of the predicted future evolution of the process state or output signal. This is accomplished by the solution of a min–max problem denoted $P(x)$

$$J^*(x) = \min_{\mathbf{v}} \max_{\mathbf{w} \in W_N} J(\mathbf{v}, \mathbf{w}, x), \tag{3}$$

where W_N denotes the set of possible disturbance sequences of length N:

$$W_N = \{\mathbf{w} |\ \|w_i\|_\infty \leqslant \varepsilon,\ i = 0, \ldots, N-1\}.$$

This optimization problem is solved at each sampling time and the solution $\mathbf{v}^*(x)$ is applied using the well known receding horizon approach [7]; i.e., only the first component of $\mathbf{v}^*(x)$ is used and the control input applied to the system is given by $u_0 = Kx + v_0^* = K_{\mathrm{MPC}}(x)$.

Taking into account (2), matrices H_x, H_u and H_w can be found (see [5,7]) in such a way that

$$J(\mathbf{v}, \mathbf{w}, x) = \|H_x x + H_u \mathbf{v} + H_w \mathbf{w}\|_2^2. \tag{4}$$

The cost function is a convex quadratic function on $\mathbf{v}$, x and $\mathbf{w}$ because it is the square of the Euclidean norm of a vector which depends linearly on these parameters (see [2]).

Function $J(\mathbf{v}, \mathbf{w}, x)$ is convex in $\mathbf{w}$, thus the maximum will be attained at least at one of the vertices $\mathbf{w}_i$ of the polyhedron W_N (see [2, Theorem 3.4.6]). The maximizer is not unique and the maximum can also be attained at another vector $\mathbf{w} \notin \mathrm{ver}(W_N)$, where $\mathrm{ver}(W_N)$ is the set of vertices of W_N. However, the maximum is unique and that is what is needed to solve the inner maximization problem (the maximizer is indeed irrelevant). The maximum of $J(\mathbf{v}, \mathbf{w}, x)$ can therefore be obtained evaluating the cost function at the set of vertices of the hypercube W_N. The min–max problem can be rewritten as

$$J^*(x) = \min_{\mathbf{v}} \max_{\mathbf{w} \in W_N} J(\mathbf{v}, \mathbf{w}, x) = \min_{\mathbf{v}} J_{\max}(\mathbf{v}, x),$$

with

$$J_{\max}(\mathbf{v}, x) = \max_{\mathbf{w} \in W_N} J(\mathbf{v}, \mathbf{w}, x) = \max_{\mathbf{w}_i \in \mathrm{ver}(W_N)} J(\mathbf{v}, \mathbf{w}_i, x). \tag{5}$$

Function $J(\mathbf{v}, \mathbf{w}, x)$ is convex on $\mathbf{v}$, and as $R > 0$, it holds that $H_u^{\mathrm{T}} H_u > 0$. Thus $J(\mathbf{v}, \mathbf{w}, x)$ is indeed strictly convex on $\mathbf{v}$. On the other hand, $J_{\max}(\mathbf{v}, x)$ is the pointwise maximum of a set of strictly convex functions of $\mathbf{v}$. Therefore $J_{\max}(\mathbf{v}, x)$ is also strictly convex on $\mathbf{v}$ [6]. This