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On controllability of diagonal systems with one-dimensional input space

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Abstract

This paper deals with diagonal systems on a Hilbert state space with a one-dimensional input space and a (possibly unbounded) control operator. A priori it is not assumed that the input operator is admissible. Necessary and sufficient conditions for different notions of controllability such as null-controllability, exact controllability and approximate controllability are presented. These conditions, which are given in terms of the eigenvalues of the diagonal operator and in terms of the control operator, are linked with the theory of interpolation in Hardy spaces.

Keywords: Controllability; Observability; Admissibility; Semigroup system; Riesz basis; Diagonal system; Carleson measure

1. Introduction

Controllability is an important property of a distributed parameter system, which has been extensively studied in the literature, see for example [7,1,13]. In this paper, we study controllability of systems whose generator has a Riesz basis of eigenvectors and a one-dimensional input space. This class might seem restrictive, but it is fairly general nevertheless, because many semigroups considered in the literature have a Riesz basis of eigenvectors, and because a practically implemented system will have a finite-dimensional input space. It has been noted in the literature that exact controllability rarely holds if the input space is one dimensional, see Triggiani [16], Rebarber and Weiss [12] and Jacob and Zwart [5], and therefore we study weaker notions such as null-controllability and approximate controllability as well. In applications, null-controllability is often sufficient. For example, the finite cost condition is implied by null-controllability and thus the existence of a unique optimal solution to various quadratic cost minimization problems is guaranteed.

On a Hilbert space H, we consider the following system:

$$\dot{x}(t) = Ax(t) + bu(t), \quad x(0) = x_0, \quad t \geqslant 0. \tag{1}$$

We assume that A is the infinitesimal generator of an exponentially stable C_0-semigroup $(T(t))_{t\geqslant 0}$ on H which possesses a sequence of normalized eigenvectors $\{\phi_n\}_{n\in\mathbb{N}}$ forming a Riesz basis for H, with associated eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$, that is,

$$A\phi_n = \lambda_n \phi_n, \quad n \in \mathbb{N}.$$

Since $(T(t))_{t\geqslant 0}$ is assumed to be exponentially stable we have $\sup_{n\in\mathbb{N}} \operatorname{Re}\lambda_n < 0$. Let ψ_n be an eigenvector of A^* corresponding to the eigenvalue $\overline{\lambda_n}$. Without loss of generality we can assume that $\langle \phi_n, \psi_n\rangle = 1$. Then the sequence $\{\psi_n\}_{n\in\mathbb{N}}$ forms a Riesz basis of H and every $x \in H$ can be written as

$$x = \sum_{n\in\mathbb{N}} \langle x, \psi_n\rangle \phi_n = \sum_{n\in\mathbb{N}} \langle x, \phi_n\rangle \psi_n.$$

For every $\alpha \in \mathbb{R}$ we introduce the interpolation space

$$H_\alpha = \left\{ \sum_{n=1}^{\infty} x_n \phi_n \mid \{x_n |\lambda_n|^\alpha\}_{n\in\mathbb{N}} \in \ell^2 \right\},$$

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equipped with the scalar product

$$\langle x, y\rangle_\alpha := \sum_{n\in\mathbb{N}} \langle x, \psi_n\rangle \overline{\langle y, \psi_n\rangle} |\lambda_n|^{2\alpha}.$$

The spaces H_α are Hilbert spaces with $H_0 = H$ and $H_1 = D(A)$. We denote the dual pairing between H_α and $H_{-\alpha}$ by $\langle \cdot, \cdot\rangle_{H_\alpha\times H_{-\alpha}}$. In the sequel let $\alpha \geqslant 0$, $b \in H_{-\alpha}$ and $u \in L^2(0,\infty)$. Thus, b can be represented by a sequence $\{b_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ with $\{b_n|\lambda_n|^{-\alpha}\}_{n\in\mathbb{N}} \in \ell^2$, that is $b_n := \langle \psi_n, b\rangle_{H_\alpha\times H_{-\alpha}}$. For more information on the spaces $H_{-\alpha}$ see for example [15]. One important feature of these interpolation spaces $H_{-\alpha}$ is that the semigroup $(T(t))_{t\geqslant 0}$ can be extended to a C_0-semigroup on $H_{-\alpha}$, which we denote by $(T_{-\alpha}(t))_{t\geqslant 0}$, and the generator of this extended semigroup, denoted by $A_{-\alpha}$, is an extension of A. By a solution of (1) we mean the so-called mild solution

$$x(t) := T(t)x_0 + \int_0^t T_{-\alpha}(t-s)bu(s)\,\mathrm{d}s, \quad t\geqslant 0. \tag{2}$$

Thus the solution is a continuous $H_{-\alpha}$-valued function. For $\tau \geqslant 0$ we introduce the operators $\mathscr{B}_\tau \in \mathscr{L}(L^2(0,\infty), H_{-\alpha})$ and $\mathscr{B}_\infty \in \mathscr{L}(L^2(0,\infty), H_{-\alpha})$ by

$$\mathscr{B}_\tau u := \int_0^\tau T_{-\alpha}(\tau - s)bu(s)\,\mathrm{d}s,$$

and

$$\mathscr{B}_\infty u := \int_0^\infty T_{-\alpha}(s)bu(s)\,\mathrm{d}s,$$

respectively. In the literature on infinite-dimensional systems it is often assumed that the operator b is admissible for the semigroup $(T(t))_{t\geqslant 0}$, and thus for some of our results we will include admissibility in the assumptions.

Definition 1.1. b is called *finite-time admissible for* $(T(t))_{t\geqslant 0}$, if there exists some $\tau > 0$ such that $\mathscr{B}_\tau u \in H$ for every $u \in L^2(0,\infty)$.

Note that admissibility implies $b \in H_{-\alpha}$ for every $\alpha > \frac{1}{2}$, see Rebarber and Weiss [12]. For exponentially stable systems the notion of finite-time admissibility is equivalent to the notion of infinite-time admissibility, that is, $\mathscr{B}_\infty u \in H$ for every $u \in L^2(0,\infty)$. We thus simply say *admissibility* instead of *finite-time* or *infinite-time admissibility*. Admissibility implies that $\mathscr{B}_\tau, \mathscr{B}_\infty \in \mathscr{L}(L^2(0,\infty), H)$ and that the mild solution of (1) corresponding to an initial condition $x(0) = x_0 \in H$ and to $u \in L^2(0,\infty)$ is a continuous H-valued function of t. For further information on admissibility we refer the reader to the survey [4]. We shall discuss the following controllability concepts.

Definition 1.2. Let $\tau > 0$. We say that system (1) is

1. *null-controllable in time* τ, if $R(T(\tau)) \subset R(\mathscr{B}_\infty)$;
2. *approximately controllable*, if $R(\mathscr{B}_\infty) \cap H$ is dense in H;
3. *exactly controllable*, if $H \subset R(\mathscr{B}_\infty)$.

Here $R(\cdot)$ denotes the range of an operator. It is easy to see that every exactly controllable system is approximately controllable and null-controllable in any time $\tau > 0$. Further, null-controllability in time τ_1 implies null-controllability in time τ_2 if $\tau_2 \geqslant \tau_1 > 0$, and in Example 2.6 we show that the notion of null-controllability can depend on τ.

Let b be an admissible control operator for $(T(t))_{t\geqslant 0}$. In [14] it is shown that system (1) is exactly controllable if and only if the system is in finite-time exactly controllable, that is, there exists $\tau > 0$ such that $H = R(\mathscr{B}_\tau)$. The following proposition shows that a similar result holds for null-controllability.

Proposition 1.3. *If b is admissible for $(T(t))_{t\geqslant 0}$, then the following statements are equivalent*:

1. *System* (1) *is null-controllable in time τ for some $\tau > 0$.*
2. *$R(T(\tau)) \subset R(\mathscr{B}_\tau)$ for some $\tau > 0$.*

The proof of this proposition will be given in Section 5.

We proceed as follows. In Section 2 we prove equivalent conditions for null-controllability, and in Section 3 we study the notion of exact controllability. Section 4 is devoted to approximate controllability. Finally, in Section 5 we present the proofs of our main results.

2. Conditions for null-controllability in time τ

In this section we derive equivalent conditions for null-controllability in time τ. We have the following main result.

Theorem 2.1. *The following statements are equivalent*:

1. *System* (1) *is null-controllable in time τ.*
2. *There exists a constant $m > 0$ such that for all $h > 0$ and all $\omega \in \mathbb{R}$*:

$$\sum_{-\lambda_n \in R(\omega,h)} \frac{|\mathrm{Re}\,\lambda_n|^2}{|b_n|^2} \mathrm{e}^{2\tau\,\mathrm{Re}\,\lambda_n} \prod_{k\neq n} \left| \frac{\overline{\lambda_n} + \lambda_k}{\lambda_n - \lambda_k} \right|^2 \leqslant mh, \tag{3}$$

where $R(\omega,h) := \{s \in \mathbb{C}_+ \mid \mathrm{Re}\,s < h,\ \omega - h < \mathrm{Im}\,s < \omega + h\}$.

3. *$\{b_n \mathrm{e}^{\lambda_n(\cdot - \tau)}\}_n$ is a Besselian basis in the closure of its span in $L^2(0,\infty)$.*

The proof of this theorem will be given in Section 5. In the following remarks we study the condition (3).

Remark 2.2. For $\beta > 0$ we consider the sequence $\{\lambda_n\}_{n\in\mathbb{N}} := \{-n^\beta\}_{n\in\mathbb{N}}$. Then for $k \geqslant n$ we have

$$\left| \frac{\overline{\lambda_n} + \lambda_k}{\lambda_k - \lambda_n} \right|^2 \approx 1 + 4\left(\frac{n}{k}\right)^\beta,$$