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Implicit Euler numerical scheme and chattering-free implementation of sliding mode systems

Vincent Acary *, Bernard Brogliato

[a] INRIA, Bipop team-project, Inovallée Montbonnot. 655, Avenue de l'Europe, 38334 Saint Ismier cedex, France

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ABSTRACT

In this paper it is shown that the implicit Euler time-discretization of some classes of switching systems with sliding modes, yields a very good stabilization of the trajectory and of its derivative on the sliding surface. Therefore the spurious oscillations which are pointed out elsewhere when an explicit method is used, are avoided. Moreover the method (an *event-capturing*, or *time-stepping* algorithm) allows for multiple switching surfaces (i.e., a sliding surface of codimension $\geqslant 2$). The details of the implementation are given, and numerical examples illustrate the developments. This method may be an alternative method for chattering suppression, keeping the intrinsic discontinuous nature of the dynamics on the sliding surfaces. Links with discrete-time sliding mode controllers are studied.

1. Introduction

Sliding mode controllers are widely used because of their intrinsic robustness properties [1,2]. Some important fields of application are induction motors [3], aircraft control [4], hard disk drives [5], solar systems [6], and autonomous robots [7]. However they are known to generate chattering which renders their application delicate. Solutions to cope with chattering or reduce its effects have been proposed, see e.g. [8–10], which also have their own limitations [10]. One drawback of these solutions is that they usually destroy the intrinsic discontinuous nature of sliding mode control. Fundamentally, these control schemes are of the switching discontinuous type and they yield closed-loop systems that can be recast into Filippov's differential inclusions. The numerical simulations of such nonsmooth dynamical systems is non trivial and it has received a lot of attention, see [11] and references therein. In this paper we focus on time-stepping methods, which have an interest not only for the sake of numerical simulation, but also for the real implementations of sliding mode controllers on discrete-time systems. Recently it has been shown that the *explicit* Euler method generates unwanted effects like spurious oscillations (also called chattering effects) around the switching surface [12,13]. In parallel, the digital implementation of sliding mode controllers has been studied in [14], where the Zero-Order Holder (ZOH) discretization is used.

The purpose of this paper is to analyze the *implicit* (*backward*) Euler method for some particular classes of differential inclusions, that include sliding mode controllers. It is shown that, besides convergence and order results, the advantage of the implicit method is that it allows one to get a very accurate and smooth stabilization on the switching surface (of codimension one or larger than one). Roughly speaking, this is due to the fact that the switches are no longer monitored by the state at step k, but by a *multiplier* (a slack variable in a nonlinear programming language). The multivalued part of the $\operatorname{sgn}(\cdot)$ function, i.e. a multifunction, is then correctly taken into account, avoiding stiff problems. The advantage of such "dual" methods in terms of their accuracy on the sliding surface has already been noticed in [15] in an event-driven context, where the motivation was the simulation of mechanical systems with Coulomb friction. From a numerical point of view, our study shows that convergence and order results may not be sufficient to guarantee that the derivative of the state is correctly approximated on the switching surface. The implicit method adapts naturally to an arbitrary large number of switching surfaces, that is not the case of most of the other methods which become quite cumbersome as soon as more than two switching surfaces are considered. A further advantage of

* Corresponding author. Tel.: +33 4 61 52 29; fax: +33 4 61 54 77.
E-mail addresses: vincent.acary@inrialpes.fr (V. Acary), bernard.brogliato@inrialpes.fr (B. Brogliato).

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the proposed method is that contrary to other methods that have been studied and which destroy the intrinsic discontinuous nature of sliding mode systems[1] (like the so-called *boundary layer control*, or various filtering techniques), our method keeps the multivalued discontinuity and consequently the fundamental aspects and properties of sliding mode control from a Filippov's system point of view. Moreover, sampling rates need not be high to reduce chattering, contrary to other discrete sliding mode controllers. A second contribution of this paper is to show that the results that hold for the backward Euler scheme, extend to ZOH discretizations of sliding mode systems.

The paper is organized as follows: Section 2 presents a motivating example for using an implicit Euler implementation of the simplest sliding mode system. In Section 3, a class of differential inclusions is introduced and existence and uniqueness results are given under the maximal monotonicity assumption. Through several examples, the Equivalent-Control-Based Sliding-Mode-Control (ECB-SMC) and the Lyapunov-based discontinuous robust control are shown to fit well within this class of differential inclusion. In Section 4, some convergence and chattering free finite-time stabilization results are given. These central results of the paper show that the implicit Euler implementation of the differential inclusion yields a chattering free convergence in finite time on the sliding surface. Section 5 is devoted to the study of Discrete-time Sliding Mode Control and the extension to ZOH discretization. Some hints on the numerical implementation of the implicit Euler scheme are given in Section 6 and the paper ends with some numerical experiments in Section 7.

Notations and definitions: Let $A \in \mathbb{R}^{n\times m}$, then $A_{\bullet i}$ is the ith column and $A_{i\bullet}$ is the ith row. The open ball of radius $r > 0$ centered at a point $x \in \mathbb{R}^n$ is denoted by $B_r(x)$. For a set of indices $\alpha \subset \{1, \dots, n\}$ and a column vector $x \in \mathbb{R}^n$, the column vector x_α will denoted the sub-vector of corresponding indices in α, that is $x_\alpha = [x_i, i \in \alpha]^T$.

2. A simple example

To start with we consider the simplest case:

$$\dot{x}(t) \in -\mathrm{sgn}(x(t)) = \begin{cases} 1 & \text{if } x(t) < 0 \\ -1 & \text{if } x(t) > 0 \\ [-1, 1] & \text{if } x(t) = 0 \end{cases}, \quad x(0) = x_0 \qquad (1)$$

with $x(t) \in \mathbb{R}$. This system possesses a unique Lipschitz continuous solution for any x_0. The backward Euler discretization of (1) reads as:

$$\begin{cases} x_{k+1} - x_k = -hs_{k+1} \\ s_{k+1} \in \mathrm{sgn}(x_{k+1}). \end{cases} \qquad (2)$$

This method converges with at least order $\frac{1}{2}$ (see Proposition 2). Let us now state a result which shows that once the iterate x_k has reached a value inside some threshold around zero for some k, then the dual variable s_{k+1} keeps its value and so does x_{k+n} for all $n \geqslant 1$.

Lemma 1. *For all $h > 0$ and $x_0 \in \mathbb{R}$, there exists k_0 such that $x_{k_0+n} = 0$ and $\frac{x_{k_0+n+1}-x_{k_0+n}}{h} = 0$ for all $n \geqslant 1$.*

Proof. The value k_0 is defined as the first time step such that $x_{k_0} \in [-h, h]$. If $x_0 \in [-h, h]$, then $k_0 = 0$. Otherwise, the solution of the time-discretization (2) is given by $x_k = x_0 - \mathrm{sgn}(x_0)kh$, $s_k = \mathrm{sgn}(x_o)$ while $x_k \notin [-h, h]$ for $k < k_0$, and $k_0 = \lceil \frac{|x(0)|}{h} \rceil - 1$. The symbol $\lceil x \rceil$ is the ceiling function which gives the smallest integer greater than or equal to x. Let us now consider that $x_{k_0} \in [-h, h]$. The only possible solution for

$$\begin{cases} x_{k_0+1} - x_{k_0} = -hs_{k_0+1} \\ s_{k_0+1} \in \mathrm{sgn}(x_{k_0+1}) \end{cases} \qquad (3)$$

is $x_{k_0+1} = 0$ and $s_{k_0+1} = \frac{x_{k_0}}{h}$. For the next iteration, we have to solve

$$\begin{cases} x_{k_0+2} = -hs_{k_0+2} \\ s_{k_0+2} \in \mathrm{sgn}(x_{k_0+2}) \end{cases} \qquad (4)$$

and we obtain $x_{k_0+2} = 0$ and $s_{k_0+2} = 0$. The same holds for all x_{k_0+n}, s_{k_0+n}, $n \geqslant 3$, redoing the same reasoning. Clearly then the terms $(x_{k_0+n+1} - x_{k_0+n})/h$ approximating the derivative are zero for any $h > 0$. □

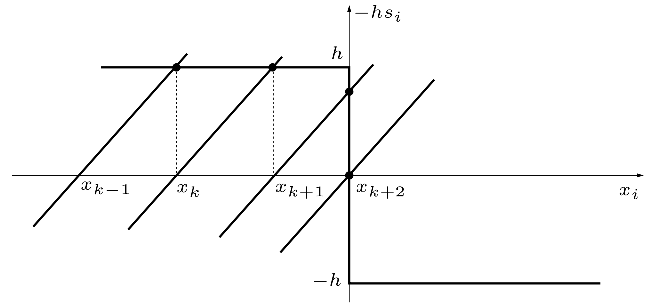

Fig. 1. Iterations of the backward Euler method.

This result is robust with respect to the numerical threshold that can be encountered in floating point operations. Indeed, let us assume that $x_{k_0} - h = \varepsilon \ll 1$, that is, $\varepsilon > 0$ is zero at the machine's precision. We obtain $s_{k_0+1} = -1$ and $x_{k_0+1} = \varepsilon$ that is zero at the machine's precision. For $n = 2$, we obtain $x_{k_0+2} = 0$ and $s_{k_0+2} = \frac{\varepsilon}{h}$. This robustness stems from the fact that the dynamics is not only monitored by the sign of x_k but also by the belongingness to the interior of $[-1, 1]$ of the "dual" variable s_{k+1}.

Consequently this result shows that there are no spurious oscillations around the switching surface, contrary to other time-stepping schemes like the explicit Euler method [12,13]. Remarkably Lemma 1 holds for any $h > 0$, which means that even a large time step assures a smooth stabilization on the sliding surface. It is noteworthy that solving the system (2) with unknown x_{k+1} and s_{k+1} is equivalent to calculate the intersection between the graph of the multivalued mapping $x_{k+1} \mapsto -h\mathrm{sgn}(x_{k+1})$ and the straight line $x_{k+1} \mapsto x_{k+1} - x_k$. This is illustrated on Fig. 1, where few iterations are depicted until the state reaches zero.

From a control perspective the input is implemented on $[t_k, t_{k+1})$ as $u_k = -\mathrm{sgn}(x_{k+1})$ as a piecewise affine function of x_k and h, where h is the sampling time. There is no problem of causality in such an implementation. It is noteworthy that in the implicit method there is absolutely no issue related to calculating $\mathrm{sgn}(0)$, or more exactly $\mathrm{sgn}(\epsilon)$ where ϵ is a very small quantity whose sign is uncertain. The implicit method automatically computes a value inside the multivalued part of the sign multifunction and may be considered as *the* time-discretization of the multifunction $\mathrm{sgn}(\cdot)$. It is easy to show that the explicit method yields an oscillation around $x = 0$, as shown in more general situations in [12,13]. Other time-stepping methods like the so-called *switched model* [11,16] fail to correctly solve the integration problem when the number of switched surfaces is too large (see also [8] for similar issues when the so-called *sigmoid blending* mechanism is implemented). Moreover this method may yield a stiff system, and from a control point of view it introduces a high-gain feedback that may not be desirable in practical applications.

[1] See [10] for a discussion on this point.