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Stability radii for positive linear time-invariant systems on time scales

T.S. Doan [a,c], A. Kalauch [a], S. Siegmund [a,*], F.R. Wirth [b]

[a] TU Dresden, Department of Mathematics and Science, Institute for Analysis, 01069 Dresden, Germany

[b] University of Würzburg, Institute for Mathematics, Am Hubland, 97074 Würzburg, Germany

[c] Hanoi Institute of Mathematics, 18 Hoang Quoc Viet Road, Hanoi, Viet Nam

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ABSTRACT

We deal with dynamic equations on time scales, where we characterize the positivity of a system. Uniform exponential stability of a system is determined by the spectrum of its matrix. We investigate the corresponding stability radii with respect to structured perturbations and show that, for positive systems, the complex and the real stability radius coincide.

1. Introduction

We consider a d-dimensional time-invariant linear system of dynamic equations

$$x^{\Delta} = Ax \tag{1}$$

($A \in \mathbb{R}^{d\times d}$) on a time scale $\mathbb{T}$, where x^{Δ} denotes the derivative of x with respect to $\mathbb{T}$. Here a *time scale* is a non-empty closed subset of $\mathbb{R}$. For the basics of the dynamic equations on time scales we refer to [1]. System (1) is said to be positive if it leaves the cone $\mathbb{R}^d_+$ invariant, i.e. if every solution starting at a point $\xi \in \mathbb{R}^d_+$ at time $t_0 \in \mathbb{T}$ remains in $\mathbb{R}^d_+$ for all times $t \in \mathbb{T}$, $t \geq t_0$. Positive systems arise in the modeling of processes where the state variables only have a meaning if they are nonnegative. For the time scales $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$ the characterization of positive systems in terms of the system matrix is well-known. We provide a characterization for positivity of system (1) on time scales.

Since a dynamical model is never an exact portrait of the real process, it is important to investigate the robustness of a stable system (1) under perturbations. We deal with uniform exponential stability, which is determined by the spectrum of the system matrix. It is of interest to find the maximal $r > 0$ such that the family of systems

$$x^{\Delta} = (A + D)x, \quad \|D\| < r, \tag{2}$$

is uniformly exponentially stable, where the matrices D are complex, real or positive, respectively. This leads to the notions of complex, real and positive stability radius. We also study the case of structured perturbations

$$A \rightsquigarrow A + BDC$$

for given structure operators B and C.

For continuous- or discrete-time systems stability radii are well-investigated notions, see [2]. A discussion on the differences between the complex and the real stability radius can be found in [3]. The complex stability radius is more easily analyzed and computed than the real one. For positive systems the situation is simpler, since the complex and the positive stability radius coincide. The continuous and discrete time cases are established in [4], resp. [5]. In [6,7] both cases are considered for more general perturbation classes. The importance of monotonic norms in the context of positive systems is pointed out in [4,5]. In the setting of Banach lattices similar results are obtained in [8]. Stability radii of finite dimensional positive continuous- and discrete-time systems have first been studied in [9]. However, in this reference the condition that B and C have to be nonnegative is not explicitly stated. An example in [7] shows that this assumption is essential.

The stability of intervals of nonnegative matrices is studied in [10,11].

* Corresponding author. Tel.: +49 351 4633 4633; fax: +49 351 4633 4664.
E-mail addresses: dtson@tu-dresden.de (T.S. Doan), anke.kalauch@tu-dresden.de (A. Kalauch), siegmund@tu-dresden.de (S. Siegmund), wirth@mathematik.uni-wuerzburg.de (F.R. Wirth).

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In the present paper we deal with positive systems on arbitrary time scales. Combining the Perron–Frobenius theory for positive matrices and for Metzler matrices, respectively, we show that for such systems the complex and the real stability radius with respect to structured perturbations coincide.

2. Preliminaries

In the following $\mathbb{K}$ denotes the real ($\mathbb{K} = \mathbb{R}$) or the complex ($\mathbb{K} = \mathbb{C}$) field. For $z \in \mathbb{C}$ and $r \in \mathbb{R}$ we define $B_r(z) = \{x \in \mathbb{C}: \|x - z\| \le r\}$. Consider on $\mathbb{C}^d$ the norm $\|x\| = (\sum_{i=1}^n |x_i|^2)^{\frac{1}{2}}$, such that one has $\|x + iy\|^2 = \|x\|^2 + \|y\|^2$ for $x, y \in \mathbb{R}^d$. As usual, $\mathbb{K}^{d\times d}$ denotes the space of square matrices with d rows, equipped with the according operator norm (spectral norm), and I_d is the identity mapping on $\mathbb{K}^d$. $\sigma(A) \subset \mathbb{C}$ denotes the set of eigenvalues of a matrix $A \in \mathbb{K}^{d\times d}$. The *spectral radius*, respectively the *spectral abscissa* of A are given by

$$\rho(A) := \max\{|\lambda|: \lambda \in \sigma(A)\} \quad \text{and} \quad \mu(A) := \max\{\Re\lambda: \lambda \in \sigma(A)\}.$$

Let $\mathbb{R}^d$ be equipped with the standard entrywise ordering, i. e. $x \le y$ if and only if $x_i \le y_i$ for all $i \in \{1, \dots, d\}$, and denote by $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d: 0 \le x\}$ the set of all nonnegative vectors. Analogously, the set of all nonnegative matrices in $\mathbb{R}^{n\times m}$ is denoted by $\mathbb{R}^{n\times m}_+$. For $A = (a_{ij})_{i,j} \in \mathbb{C}^{n\times m}$ we define $|A| := (|a_{ij}|)_{i,j}$, so that $|A|$ denotes the matrix obtained by taking the absolute value entrywise.

A *time scale* $\mathbb{T}$ is a non-empty, closed subset of the reals $\mathbb{R}$. For the purpose of this paper we assume from now on that $\mathbb{T}$ is unbounded from above, i.e. $\sup \mathbb{T} = \infty$. On $\mathbb{T}$ the *graininess* is defined by

$$\mu^*(t) := \inf\{s \in \mathbb{T}: t < s\} - t.$$

A point $t \in \mathbb{T}$ is called *right-dense* if $\mu^*(t) = 0$ and *right-scattered* if $\mu^*(t) > 0$. Similarly, $t \in \mathbb{T}$ is called *left-dense* if $t - \sup\{s \in \mathbb{T}: s < t\} = 0$ and *left-scattered* if $t - \sup\{s \in \mathbb{T}: s < t\} > 0$. For a function $f: \mathbb{T} \to \mathbb{K}$ and a point $t_0 \in \mathbb{T}$ we say that $f^\Delta(t_0) \in \mathbb{K}$ is the *derivative* of f in t_0 if for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $t \in (t_0 - \delta, t_0 + \delta) \cap \mathbb{T}$ the inequality

$$\left|f(t_0 + \mu^*(t_0)) - f(t) - (t_0 + \mu^*(t_0) - t)f^\Delta(t_0)\right| \le \varepsilon \left|t_0 + \mu^*(t_0) - t\right|$$

is satisfied. If t_0 is right-scattered, one obtains

$$f^\Delta(t_0) = \frac{f(t_0 + \mu^*(t_0)) - f(t_0)}{\mu^*(t_0)}. \tag{3}$$

An introduction into dynamic equations on time scales can be found in [1].

For $A \in \mathbb{K}^{d\times d}$ we consider the d-dimensional linear system of dynamic equations on the time scale $\mathbb{T}$

$$x^\Delta = Ax. \tag{4}$$

We recall the classical examples for this setup.

Example 1. If $\mathbb{T} = \mathbb{R}$ we have a linear time-invariant system of the form $\dot{x}(t) = Ax(t)$. If $\mathbb{T} = h\mathbb{Z}$, then (4) reduces to $(x(t + h) - x(t))/h = Ax(t)$ or, equivalently, to $x(t + h) = [I_d + hA]x(t)$.

Let $e_A: \{(t, \tau) \in \mathbb{T} \times \mathbb{T}: t \ge \tau\} \to \mathbb{K}^{d\times d}$ denote the *transition matrix* corresponding to (4), that is, $x(t) = e_A(t, \tau)\xi$ solves the initial value problem (4) with initial condition $x(\tau) = \xi$ for $\xi \in \mathbb{K}^d$ and $t, \tau \in \mathbb{T}$ with $t \ge \tau$. Due to (3), for a right-scattered point $t_0 \in \mathbb{T}$ we have

$$e_A(t_0 + \mu^*(t_0), t_0) = I_d + \mu^*(t_0)A. \tag{5}$$

For a scalar system $x^\Delta = \lambda x$ ($\lambda \in \mathbb{C}$, $1 + \mu^*(t)\lambda \ne 0$ for all $t \in \mathbb{T}$) one obtains

$$|e_\lambda(t, \tau)| = \exp\left(\int_\tau^t \lim_{s \searrow \mu^*(u)} \frac{\log|1 + s\lambda|}{s}\, \Delta u\right), \tag{6}$$

cf. [1, Theorems 2.33 and 2.35].

The subsequent notions are recalled from [12].

Definition 2 (*Exponential Stability*). Let $\mathbb{T}$ be a time scale which is unbounded above. We call system (4)

(i) *exponentially stable* if there exists a constant $\alpha > 0$ such that for every $s \in \mathbb{T}$ there exists $K(s) \ge 0$ with

$$\|e_A(t, s)\| \le K(s) \exp(-\alpha(t - s)) \quad \text{for } t \ge s,$$

(ii) *uniformly exponentially stable* if K can be chosen independently of s in the definition of exponential stability.

Observe that $K(s) \ge 1$ follows from the definition for $t = s$.

In general, exponential stability does not imply uniform exponential stability [12]. The existence of a uniformly exponentially stable system can only be guaranteed if the time scale $\mathbb{T}$ has bounded graininess [13, Theorem 3.1].

In [13, Example 4.1] it is shown that exponential stability of (4) cannot be characterized by the spectrum of its matrix, whereas uniform exponential stability is determined by the spectrum. Note that although the following proposition is only proved for a real matrix in [13, Theorem 3.2] the statement remains true for an arbitrary complex matrix without any modification in the proof.

Proposition 3 (*[13, Theorem 3.2]*). *For $A \in \mathbb{K}^{d\times d}$ system (4) is uniformly exponentially stable if and only if system*

$$x^\Delta = \lambda x \tag{7}$$

is uniformly exponentially stable for every $\lambda \in \sigma(A)$.

Since we want to consider stability radii with respect to uniform exponential stability, we denote

$$\mathcal{US}_\mathbb{C}(\mathbb{T}) = \{\lambda \in \mathbb{C}: \text{system (7) is uniformly exponentially stable}\}.$$

So, for $A \in \mathbb{K}^{d\times d}$ system (4) is uniformly exponentially stable if and only if $\sigma(A) \subset \mathcal{US}_\mathbb{C}(\mathbb{T})$.

Remark 4. (i) Since uniform exponential stability is robust it follows that $\mathcal{US}_\mathbb{C}(\mathbb{T})$ is an open set [13, Proposition 3.1].

(ii) For any $h \ge \max\{\mu^*(t): t \in \mathbb{T}\}$ the system

$$x^\Delta = \frac{-1}{2h}x$$

is uniformly exponentially stable [13, Proof of Theorem 3.1]. On the other hand, for any $\alpha > 0$ the system

$$x^\Delta = \alpha x$$

is not exponentially stable. Therefore, 0 is contained in the boundary of $\mathcal{US}_\mathbb{C}(\mathbb{T})$.

(iii) Consider a scalar system (7). If there is $t_0 \in \mathbb{T}$ such that $1 + \mu^*(t_0)\lambda = 0$, then $x(t) = 0$ for all $t \in \mathbb{T}$, $t \ge t_0$, in particular (7) is uniformly exponentially stable (which follows directly from (5)). Such systems are called non-regressive, cf. [1, Definition 2.32].

In a particular case the notions of exponential stability and uniform exponential stability coincide. We call a time-scale *periodic* if there exists a constant $p > 0$ such that for every $t \in \mathbb{R}$ we have $t \in \mathbb{T}$ if and only if $t + p \in \mathbb{T}$. In this case p is called a *period* of the time-scale. Clearly, if a time scale is only given as a subset of $[a, \infty)$ and satisfies a periodicity condition there, it may be extended to a periodic time scale that is unbounded above and below.

The following proposition links the results in [13,12] and will be useful in the discussion of examples below.