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Robustness of general decay stability of nonlinear neutral stochastic functional differential equations with infinite delay

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ABSTRACT

This paper establishes the existence-and-uniqueness theorem of neutral stochastic functional differential equations with infinite delay and examines the almost sure stability of this solution with general decay rate. This result may be used to examine almost sure robust stability. To illustrate our idea more carefully, we carefully discuss a scalar stochastic integro-differential equation with neutral type and its asymptotic stability, including the exponential stability and the polynomial stability.

1. Introduction

In this paper, we consider the neutral stochastic functional differential equation with infinite delay

$$\mathrm{d}[x(t) - u(t, x_t)] = f(t, x_t)\mathrm{d}t + g(t, x_t)\mathrm{d}w(t), \tag{1.1}$$

where $u : \mathbb{R}_+ \times BC((-\infty, 0]; \mathbb{R}^n) \to \mathbb{R}^n$ is a continuous functional, $f : \mathbb{R}_+ \times BC((-\infty, 0]; \mathbb{R}^n) \to \mathbb{R}^n$ and $g : \mathbb{R}_+ \times BC((-\infty, 0]; \mathbb{R}^n) \to \mathbb{R}^{n\times m}$ are Borel measurable, $w(t)$ is an m-dimensional Brownian motion. Eq. (1.1) may be regarded as the stochastically perturbed system of the deterministic neutral functional differential equation with infinite delay

$$\frac{\mathrm{d}[x(t) - u(t, x_t)]}{\mathrm{d}t} = f(t, x_t). \tag{1.2}$$

In general, time delay and system uncertainty are commonly encountered and are often sources of instability (see [1]). It is therefore interesting to consider the stability of Eq. (1.1), whose delay is infinite. Under the assumption that system (1.2) is asymptotically stable, this paper determines how much stochastic perturbation this system can tolerate without losing the property of asymptotic stability. Such a point of view is described as the problem of robust stability, which has received a great deal of attention in recent years. Haussmann [2] and Ichikawa [3] studied the robustness of stability for a linear system with a linear stochastic perturbation. Mao [4] investigated the linear stochastic perturbation with delay of this linear equation. In [5], Mao established the robust stability of the nonlinear differential system with a stochastic functional perturbation. There is also an extensive literature concerned with the robust stability of stochastic integro-differential equations and here we only mention [6–8]. However, little is as yet known about the robust stability of functional differential systems with neutral type although these neutral systems possess a large number of applications in applied mathematics and engineering (see [9,10]).

Stability constitutes a central characteristic of the behavior of a (deterministic or stochastic) dynamical system. There exists an extensive literature on exponential stability of stochastic differential equations, for example, [11–14,10,15,16]. There is much literature concerning polynomial stability of stochastic differential equations, and we mention here only [17,18]. These different stabilities show that the speed with which the solution decays to zero is different. Then these stability concepts are generalized to the stability with general decay rate (see [19,20,10,8]).

To consider the asymptotic stability with general decay rate, let us introduce the following ψ-type function, which will be used as the decay function in this paper.

Definition 1.1. The function $\psi : \mathbb{R} \to (0, \infty)$ is said to be the ψ-type function if this function satisfies the following conditions:

(i) it is continuous and nondecreasing in $\mathbb{R}$ and differentiable in $\mathbb{R}_+$;

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(ii) $\psi(0) = 1$ and $\psi(\infty) = \infty$;
(iii) $\phi = \sup_{t\geq 0}[\psi'(t)/\psi(t)] < \infty$;
(iv) for any $t, s \geq 0$, $\psi(t) \leq \psi(s)\psi(t-s)$.

It is obvious that functions $\psi(t) = e^{\alpha t}$ and $\psi = (1+t^+)^{\bar{\alpha}}$ for all $\alpha, \bar{\alpha} > 0$ are ψ-type functions since they satisfy the above four conditions.

In this paper, we will show that Eq. (1.1) has the following properties:

- This equation almost surely admits a global solution.
- The solution of this equation is almost surely ψ-type stable, namely,

$$\limsup_{t\to\infty} \frac{\log|x(t,\xi)|}{\log\psi(t)} < 0 \quad \text{a.s.} \tag{1.3}$$

It is obvious that this ψ-type stability implies the exponential stability and polynomial stability when $\psi(t) = e^{\alpha t}$ and $\psi = (1+t^+)^{\bar{\alpha}}$ for any $\alpha, \bar{\alpha} > 0$, respectively.

In the next section, we give some necessary notation and lemmas. Then we establish a general existence-and-uniqueness theorem of the global solution to Eq. (1.1) and examine almost sure ψ-type stability of this solution in Section 3 and then we imposes some conditions on coefficients u, f and g under which we furthermore examine this asymptotic ψ-type stability of this solution. These conditions show the robustness of this almost sure ψ-type stability. In Section 4, we examine the scalar neutral stochastic integro-differential equation with neutral type. By choosing $\psi(t) = e^t$ and $\psi = \log(1+t^+)$, we consider the exponential stability and the polynomial stability of this equation and analyze robustness of these stabilities.

2. Preliminaries

Throughout this paper, unless otherwise specified, we use the following notation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $w(t)$ be an m-dimensional Brownian motion defined on this probability space. If $x(t)$ is an $\mathbb{R}^n$-valued stochastic process on $t \in \mathbb{R}$, define $x_t = x_t(\theta) := \{x(t+\theta) : -\infty < \theta \leq 0\}$ for $t \geq 0$ and let $\tilde{x}(t) := x(t) - u(t, x_t)$.

Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^n$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, denote its trace norm by $|A| = \sqrt{\text{trace}(A^TA)}$. Let $\mathbb{R}_+ = [0, \infty)$. Denote by $C((-\infty, 0]; \mathbb{R}^n)$ the family of continuous functions from $(-\infty, 0]$ to $\mathbb{R}^n$. Similarly, denote by $BC((-\infty, 0]; \mathbb{R}^n)$ the family of bounded continuous functions from $(-\infty, 0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{\theta\leq 0}|\varphi(\theta)| < \infty$, which forms a Banach space. If $a, b \in \mathbb{R}$, $a \wedge b$ denotes the minimum of a and b and $a \vee b$ represents their maximum. Let $a^+ = a \vee 0$.

Let $C^{1,2}(\mathbb{R}^n; \mathbb{R}_+)$ denote the family of all nonnegative functions $V(t, x)$ on $\mathbb{R}_+ \times \mathbb{R}^n$ which are continuously differentiable in t and twice differentiable in x, and define an operator $\mathcal{L}V : \mathbb{R}_+ \times BC((-\infty, 0]; \mathbb{R}^n) \to \mathbb{R}$ by

$$\mathcal{L}V(t,\varphi) = V_t(t, \varphi(0)) + V_x(t, \varphi(0) - u(t,\varphi))f(t,\varphi) + \frac{1}{2}\text{trace}[g^T(t,\varphi)V_{xx}(t, \varphi(0) - u(t,\varphi))g(t,\varphi)], \tag{2.1}$$

where

$$V_x(t,x) = \left(\frac{\partial V(t,x)}{\partial x_1}, \frac{\partial V(t,x)}{\partial x_2}, \ldots, \frac{\partial V(t,x)}{\partial x_n}\right),$$

$$V_{xx}(t,x) = \left[\frac{\partial^2 V(t,x)}{\partial x_i \partial x_j}\right]_{n\times n}.$$

Let us emphasize that $\mathcal{L}V$ is a functional defined on $\mathbb{R}_+ \times BC((-\infty, 0]; \mathbb{R}^n)$ while V is a function on $\mathbb{R}_+ \times \mathbb{R}^n$.

Let $L^p((-\infty, 0]; \mathbb{R}^n)$ denote all functions $h : (-\infty, 0] \to \mathbb{R}^n$ such that $\int_{-\infty}^0 |h(s)|^p ds < \infty$. We give the following lemma.

Lemma 2.1. *Let $\varphi \in BC((-\infty, 0]; \mathbb{R}^n) \cap L^p((-\infty, 0]; \mathbb{R}^n)$ for any $p > 0$. Then for any $q > p$, $\varphi \in L^q((-\infty, 0]; \mathbb{R}^n)$.*

This proof is trivial, so we omit it.

Let $\mathcal{M}_0$ denote all probability measures μ on $(-\infty, 0]$. For any $\varepsilon \geq 0$, define

$$\mathcal{M}_\varepsilon := \left\{\mu \in \mathcal{M}_0; \mu_\varepsilon := \int_{-\infty}^0 \psi^\varepsilon(-\theta)d\mu(\theta) < \infty\right\}. \tag{2.2}$$

Clearly, μ_ε holds the following nice property:

Lemma 2.2. *Fix $\varepsilon_0 > 0$. For any $\varepsilon \in [0, \varepsilon_0]$, μ_ε is continuously nondecreasing and satisfies $\mu_{\varepsilon_0} \geq \mu_\varepsilon \geq \mu_0 = 1$ and $\mathcal{M}_{\varepsilon_0} \subseteq \mathcal{M}_\varepsilon \subseteq \mathcal{M}_0$.*

Proof. For any $\theta \in (-\infty, 0]$, $\psi^\varepsilon(-\theta)$ is nondecreasing in ε. This implies that μ_ε is nondecreasing in ε and hence $\mu_{\varepsilon_0} \geq \mu_\varepsilon \geq \mu_0 = 1$ and $\mathcal{M}_{\varepsilon_0} \subseteq \mathcal{M}_\varepsilon \subseteq \mathcal{M}_0$ since $\varepsilon \in [0, \varepsilon_0]$. The dominated convergence Theorem (see [21, Theorem 1.21, P11]) gives continuity. □

For the purpose of stability, assume that $f(t, 0) = g(t, 0) = u(t, 0) = 0$. This implies that Eq. (1.1) admits a trivial solution $x(t, 0) \equiv 0$. We also impose the following standing assumptions on coefficients u, f and g:

Assumption 2.1. There exist a constant $\kappa \in (0, 1)$ and a probability measure $\eta \in \mathcal{M}_\varepsilon$ for any given $\varepsilon > 0$ such that

$$|u(t,\varphi) - u(t,\phi)| \leq \kappa \int_{-\infty}^0 |\varphi(\theta) - \phi(\theta)|d\eta(\theta)$$

on $t \geq 0$ for any $\varphi, \phi \in BC((-\infty, 0]; \mathbb{R}^n)$.

Assumption 2.2. Both f and g satisfy the local Lipschitz condition, that is, for any $k > 0$, there exists a c_k such that

$$|f(t,\varphi) - f(t,\phi)| \vee |g(t,\varphi) - g(t,\phi)| \leq c_k\|\varphi - \phi\|$$

on $t \geq 0$ for those $\varphi, \phi \in BC((-\infty, 0]; \mathbb{R}^n)$ with $\|\varphi\| \vee \|\phi\| \leq k$.

Let us present two useful lemmas.

Lemma 2.3. *Let $0 \leq \bar{\alpha} < \alpha$, $a > 0$, $b > \bar{b} \geq 0$. If*

$$a > r\bar{b}, \tag{2.3}$$

where $r = (\alpha - \bar{\alpha})(\bar{\alpha}^{\bar{\alpha}}/(\alpha^\alpha)^{1/(\alpha-\bar{\alpha})})$ when $\bar{\alpha} > 0$ and $r = 1$ when $\bar{\alpha} = 0$, then there exists a constant $\bar{a} \in (0, a)$ such that for any $t \geq 0$

$$a + bt^\alpha - \bar{b}t^{\bar{\alpha}} \geq \bar{a}. \tag{2.4}$$

Proof. If we can prove that there exists a constant $a_0 = a - \bar{a} \in (0, a)$ such that for all $t \geq 0$,

$$F(t) := a_0 + bt^\alpha - \bar{b}t^{\bar{\alpha}} \geq 0, \tag{2.5}$$

we can obtain the desired assertion. When $\bar{\alpha} = 0$, noting that $a > \bar{b}$, choosing a_0 sufficiently near a such that $a_0 > \bar{b}$, the desired result will follow. When $\bar{\alpha} > 0$, it is obvious that there exists a unique $t_0 = (\bar{\alpha}\bar{b}/\alpha b)^{1/(\alpha-\bar{\alpha})}$ such that $F'(t_0) = 0$. Under the condition $b > \bar{b} \geq 0$, by condition (2.3), choose a_0 sufficiently near a such that

$$F(t_0) = a_0 - \frac{\bar{b}(\alpha - \bar{\alpha})}{\alpha}\left(\frac{\bar{\alpha}\bar{b}}{\alpha b}\right)^{\frac{\bar{\alpha}}{\alpha-\bar{\alpha}}} > 0.$$

Noting that $F(0) = a_0 > 0$ and $F(\infty) = \infty$, for any $t \geq 0$, we have $F(t) \geq a_0 \wedge F(t_0) > 0$, as required. □

Then we gives the continuous semimartingale convergence theory (cf. [22,14]).