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Local ISS of large-scale interconnections and estimates for stability regions

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ABSTRACT

We consider interconnections of locally input-to-state stable (LISS) systems. The class of LISS systems is quite large, in particular it contains input-to-state stable (ISS) and integral input-to-state stable (iISS) systems.

Local small-gain conditions both for LISS trajectory and Lyapunov formulations guaranteeing LISS of the composite system are provided in this paper. Notably, estimates for the resulting stability region of the composite system are also given. This in particular provides an advantage over the linearization approach, as will be discussed.

1. Introduction

In this paper we study local stability properties of interconnected nonlinear systems. One of the most popular frameworks for such interconnections is input-to-state stability (ISS) introduced in [1]. This notion has been used successfully for the investigation of continuous and discrete time systems, systems with time delays, and hybrid systems. In particular the first small-gain stability condition for a feedback interconnection of two ISS systems which were given in terms of ordinary differential equations was derived in [2]. A corresponding construction of an ISS Lyapunov function for feedback interconnections has been given in [3]. These results were extended for the case of an interconnection of $n \geq 2$ systems in [4–6], respectively. Small-gain theorems for hybrid systems can be found in [7,8]. Interconnected systems with time delays have been studied in the ISS framework in [9]. A small-gain theorem for interconnections of a more general type of systems that do not satisfy the classical semigroup property has been developed in [10].

In some applications the ISS property can be rather restrictive. A less restrictive property is for example the integral input-to-state stability (iISS) property [11]. The set of iISS systems contains ISS systems as a proper subset. Small-gain theorems for interconnections of iISS systems can be found in [12,13]. Another way to weaken the ISS property is to consider its local version, local input-to-state stability (LISS), but see also [14–16] for different local stability properties. It turns out that LISS constitutes an even bigger class of nonlinear systems than iISS systems (cf. [17, Theorem 1]: iISS implies 0-GAS and [16, Lemma I.1]: 0-GAS implies LISS). In broad terms, a system is LISS if the ISS property holds locally with respect to inputs and initial states. Systems with such restrictions and a corresponding small-gain condition for feedback interconnections of two systems have been discussed in [2]. Large-scale interconnections of such systems have been considered in [18] for the first time.

Provided that the stability regions of allowable inputs and initial conditions are quantified and suitably large, LISS is a rather interesting property from an application perspective, as it allows to estimate transient and asymptotic behavior of solutions of nonlinear systems in a well-understood framework.

This paper is devoted to stability investigations of large-scale interconnected nonlinear systems. To this extent, we consider $n \geq 2$ subsystems given by

$$\dot{x}_i = f_i(x_1, \dots, x_n, u_i), \quad i = 1, \dots, n, \tag{1}$$

where $x_i \in \mathbb{R}^{N_i}$, $u_i \in \mathbb{R}^{M_i}$, and $f_i : \mathbb{R}^{\sum_j N_j + M_i} \to \mathbb{R}^{N_i}$, $i = 1, \dots, n$, are assumed to be continuous and locally Lipschitz in x uniformly for u_i in compact sets, which guarantees existence (at least on small time intervals) and uniqueness of solution for each of the systems. Let x^T denote the transposition of a vector. Introducing $x^T = (x_1^T, \dots, x_n^T) \in \mathbb{R}^N$, $N = \sum_{i=1}^n N_i$, $M = \sum_{i=1}^n M_i$, $u^T = (u_1^T, \dots, u_n^T)$, $f(x,u)^T = (f_1(x,u_1)^T, \dots, f_n(x,u_n)^T)$ we consider

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this interconnection as one composite system of a larger dimension N,

$$\dot{x} = f(x, u). \tag{2}$$

Our main results are small-gain theorems that provide sufficient conditions for the stability of such interconnections: Under the assumption that each system (1) is LISS (see below) and a small-gain condition, we show that the composite system (2) is also LISS.

In particular, we provide a local small-gain condition, which turns out to be similar but weaker than its global counterpart in [4–6]. We also show how the Lyapunov functions of the subsystems can be aggregated to a composite Lyapunov function. The approach is similar and heavily inspired by its global counterpart; however, a number of technical modifications are in order, which will be provided. Most notably and in contrast to previous works and existing literature based on linearization, our results provide estimates on the regions where the stability results hold. In addition, by utilizing the concept proposed in [16], our results also apply to stability with respect to sets, rather than just equilibrium points.

The paper is organized as follows. The next section introduces the necessary notions and formally states the problem. In Section 3 we recall corresponding global results for the stronger ISS property. Our local small-gain condition is introduced in Section 4 where we also prove some auxiliary results related to this condition. Section 5 contains the main results of the paper. In Section 5.3 we briefly highlight the advantages of LISS compared to linearization approaches. An illustrative example is considered in Section 6. Section 7 concludes the paper.

2. Notation and definitions, problem formulation

2.1. Notation

Let $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_i \geq 0,\ i = 1, \dots, n\}$ denote the positive orthant in $\mathbb{R}^n$. For $a, b \in \mathbb{R}^n_+$ let $a \ll b$ denote that $a_i < b_i$ for all $i = 1, \dots, n$ and $a \leq b$ denote $a_i \leq b_i$ for all $i = 1, \dots, n$. We write $a < b$ iff $a \leq b$ and $a \neq b$. With respect to this partial order, the minimum and maximum of two or more vectors is taken component-wise. For a vector $a \in \mathbb{R}^n$ by $|a|$ we denote the vector $(|a_1|, \dots, |a_n|)^T \in \mathbb{R}^n_+$. Observe that $|a| = \max\{a, -a\}$. The logical negation of the relation $\geq$ is denoted by $a \not\geq b$ and it means that there is at least one $i \in \{1, \dots, n\}$ such that $a_i < b_i$. It is *not* the same as the relation $<$. For $a, b \in \mathbb{R}^n_+$ we write $[a, b] := \{s \in \mathbb{R}^n_+ : a \leq s \leq b\}$, $(a, b) := \{s \in \mathbb{R}^n_+ : a < s < b\}$, and similarly $[a, b)$, $(a, b]$ to denote order intervals in $\mathbb{R}^n_+$. By $\|x\|$ we denote the Euclidean norm of $x \in \mathbb{R}^n$ and by $\|u\|_{L_\infty(T)} = \text{ess.}\sup_{t\in T} \|u(t)\|$ we denote the essential supremum norm of a measurable function u. Reference to the time interval T is usually omitted in the case $T = \mathbb{R}_+$. The set of all measurable and essentially bounded functions is denoted by L_∞. By $B(x, r)$ we denote the open ball with respect to the Euclidean norm around x of radius r. Let $\mathcal{A}$ be a nonempty set in $\mathbb{R}^n$. Then by $\|x\|_\mathcal{A} = d(x, \mathcal{A}) = \inf_{y\in\mathcal{A}} \|x - y\|$ we denote the distance between x and $\mathcal{A}$, cf. [16]. The induced L_∞-distance is denoted by $\|x\|_{L^\mathcal{A}_\infty(T)} := \text{ess.}\sup_{t\in T} \|x(t)\|_\mathcal{A}$.

A continuous operator $A : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is called monotone, if $r \leq s$ implies $A(r) \leq A(s)$. For a vector $x \in \mathbb{R}^n_+$ we denote by $x|_I$ the vector in $\mathbb{R}^n_+$ with elements

$$(x|_I)_i = \begin{cases} x_i & \text{if } i \in I \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$ if it is continuous, increasing and $\gamma(0) = 0$. It is of class $\mathcal{K}_\infty$ if, in addition, it is unbounded. We will frequently use the class $\mathcal{K}_\infty$ notation for functions that are defined only on bounded intervals $[0, r]$. In this case the function will obviously be bounded; however, it can always be extended to a $\mathcal{K}_\infty$ function on $[0, \infty)$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if, for each fixed t, the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ and, for each fixed s, the function $\beta(s, \cdot)$ is non-increasing and tends to zero at infinity.

2.2. Local input-to-state stability (LISS)

The concept of input-to-state stability (ISS) has been first introduced in [1]. Its local version, also with respect to a nonempty, compact set $\mathcal{A}$, has first appeared in [16].

Throughout let $\mathcal{A} \subset \mathbb{R}^N$ be nonempty, compact, and *zero-invariant* with respect to (2), i.e., $x(t, \xi, \underline{0}) \in \mathcal{A}$ for all $t \geq 0, \xi \in \mathcal{A}$, where $\underline{0}$ denotes the input which is identically zero and $x()$ denotes the unique solution to (2).

Definition 2.1. System (2) is *locally input-to-state stable* (LISS) with respect to $\mathcal{A}$, if there exist $\rho^0 > 0$, $\rho^u > 0$, $\gamma \in \mathcal{K}_\infty$, and $\beta \in \mathcal{KL}$, such that for all $\|\xi\|_\mathcal{A} \leq \rho^0$, $\|u\|_{L_\infty} \leq \rho^u$

$$\|x(t, \xi, u)\|_\mathcal{A} \leq \beta(\|\xi\|_\mathcal{A}, t) + \gamma(\|u\|_{L_\infty}), \quad \forall\, t \geq 0. \tag{3}$$

Here γ is called *LISS gain*.

If $\rho^0 = \rho^u = \infty$, then system (2) is called *input-to-state stable* (ISS) with respect to $\mathcal{A}$. It is known that ISS defined this way is equivalent to the existence of an ISS Lyapunov function. Here we give the definition of a LISS Lyapunov function:

Definition 2.2. A smooth function $V : \mathcal{D} \to \mathbb{R}_+$, with $\mathcal{D} \subset \mathbb{R}^N$ open, is a *LISS Lyapunov function* of (2) if there exist $\rho^0 > 0$, $\rho^u > 0$, $\psi_1, \psi_2 \in \mathcal{K}_\infty$, $\gamma \in \mathcal{K}_\infty$, and a positive definite function α such that $B(0, \rho^0) \subset \mathcal{D}$ and

$$\psi_1(\|x\|_\mathcal{A}) \leq V(x) \leq \psi_2(\|x\|_\mathcal{A}), \quad \forall\, x \in \mathbb{R}^N, \tag{4}$$

$$V(x) \geq \gamma(\|u\|) \implies \nabla V(x) \cdot f(x, u) \leq -\alpha(V(x)), \tag{5}$$

for all $\|x\|_\mathcal{A} \leq \rho^0$, $\|u\| \leq \rho^u$. The function γ is called *LISS Lyapunov gain*. If $\rho^0 = \rho^u = \infty$ then V is called an *ISS Lyapunov function*.

A related and strictly weaker stability concept (just think of the scalar system $\dot{x} = 0$) is that of local stability:

Definition 2.3. System (2) is *locally stable* (LS) with respect to $\mathcal{A}$, if there exist $\rho^0 > 0$, $\rho^u > 0$, $\sigma, \gamma \in \mathcal{K}_\infty$, such that for all $\|\xi\|_\mathcal{A} \leq \rho^0$, $\|u\|_{L_\infty} \leq \rho^u$

$$\text{ess.}\sup_{t\geq 0} \|x(t, \xi, u)\|_\mathcal{A} \leq \sigma(\|\xi\|_\mathcal{A}) + \gamma(\|u\|_{L_\infty}). \tag{6}$$

Also related is the concept of asymptotic gains.

Definition 2.4. System (2) has the *local asymptotic gain property* (LAG) with respect to $\mathcal{A}$, if there exist $\rho^0 > 0$, $\rho^u > 0$, $\gamma \in \mathcal{K}_\infty$, such that for all $\|\xi\|_\mathcal{A} \leq \rho^0$, $\|u\|_{L_\infty} \leq \rho^u$

$$\limsup_{t\to\infty} \|x(t, \xi, u)\|_\mathcal{A} \leq \gamma(\|u\|_{L_\infty}). \tag{7}$$

Note that inequality (7) is equivalent to

$$\limsup_{t\to\infty} \|x(t, \xi, u)\|_\mathcal{A} \leq \gamma(\text{ess.}\limsup_{t\to\infty} \|u\|). \tag{8}$$

In all of the above stability definitions, the reference to $\mathcal{A}$ is usually omitted when $\mathcal{A} = \{0\}$.