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Global convergence of nonlinear cascade flows with Morse–Bott zero dynamics[☆]

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ABSTRACT

We derive a sufficient condition that a flow captures the dynamics on an invariant submanifold. This leads to a refinement of the LaSalle invariance principle. As a consequence, we generalize a well-known global asymptotic stability result of nonlinear cascade systems to show global convergence to a compact invariant set. This includes the case where a globally asymptotically stable system is coupled to a Morse–Bott flow.

1. Introduction

Convergence of nonlinear cascade systems is a challenging topic, with interesting links to areas such as e.g. persistent population dynamics in mathematical biology [1], convergent activation dynamics of neural networks [2], asymptotic behavior of time-varying nonlinear differential equations [3] and feedback stabilization problems in nonlinear control theory [4]. The simplest type of a cascade flow consists of two nonlinear differential equations with a coupling structure as

$$\dot{x} = f(x, y) \tag{1}$$

$$\dot{y} = g(y). \tag{2}$$

Here the *driving dynamics* (2) evolve on Euclidean space $\mathbb{R}^m$, with $0 \in \mathbb{R}^m$ as a globally asymptotically stable equilibrium point. The *controlled dynamics* (1) are assumed to evolve on a smooth n-dimensional manifold M, such that the *zero dynamics*

$$\dot{x} = f(x, 0) \tag{3}$$

globally converge to a compact invariant subset Λ of M. In such a situation it is not guaranteed a priori that the cascaded flow (1) and (2) will also converge to $\Lambda \times \{0\}$. In fact, without further assumptions, such a persistence property of the dynamics will not hold.

The easiest and best understood situation is that when both the driving dynamics (2) and the zero dynamics (3) globally converges to asymptotically stable equilibrium points x_0 and y_0, respectively; see e.g. [5,6,4]. Then the composed system (1) and (2) is locally asymptotically stable, but may still fail to be globally asymptotically stable. A simple example that exhibits this loss of global asymptotic stability is the system

$$\dot{x} = x(x^2y^2 - 1) \tag{4}$$

$$\dot{y} = -y, \tag{5}$$

which is locally asymptotically stable around $(0, 0)$ but has $x^2y^2 = 2$ as another invariant manifold (on which solutions go to infinity) [4,7]. The main difficulty in establishing global convergence for such flows is to prove, that each trajectory of (1) and (2) is forward bounded. This is immediately implied by the well-known (and restrictive) BIBO condition, i.e. by the property that the solutions of $\dot{x} = f(x, u)$ are forward bounded for every bounded input function u. Once this condition is satisfied, global convergence of (1) and (2) to the equilibrium point (x_0, y_0) holds.

In several applications the zero dynamics are not globally convergent to a single equilibrium point, so that the existing convergence theory is not immediately applicable. Specifically, in this paper, we are interested in extending the theory to the more general situation, where the zero dynamics (3) is known to converge to a possibly infinite set $\Lambda \subset M$ of equilibria. Here, even if the solutions are forward bounded, two new phenomena arise, which are not present in the simple asymptotically stable type of systems discussed above. First, to ensure convergence of the cascade flow to $\Lambda \times \{0\}$ requires additional assumptions.

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For example, just checking pointwise convergence of the zero dynamics to single equilibrium points is not enough to guarantee the global convergence of (1) and (2). The key difficulty here is in the potential existence of homoclinic orbits or, more generally, of cyclic trajectory paths of the zero dynamics, as their presence may destroy global convergence of the composed system. The example

$$\dot{x} = x(1 - \sqrt{x^2 + y^2}) - y(b|y| + z^2) \quad (6)$$

$$\dot{y} = y(1 - \sqrt{x^2 + y^2}) + x(b|y| + z^2) \quad (7)$$

$$\dot{z} = -cz \quad (8)$$

of a Lipshitz-continuous vector field by [3] (see [8] for a smooth variant) demonstrates this phenomenon very clearly, i.e. the loss of global convergence if a non-cycle condition on the invariant sets is violated. The main point to note here is that the flow on $S^1 \times 0$ is cyclic, since the upper part of the unit circle flows to $(-1, 0)$, while the lower part converges to $(1, 0)$. In particular, for $b > 2c > 0$, $z \neq 0$, $(x, y) \neq (0, 0)$, the solutions can be shown to converge to the entire unit circle $S^1 \times 0$, while the flow on $S^1 \times 0$ converges to one of the two equilibria $(1, 0)$, $(-1, 0)$. Thus the ω-limit sets of points that start off the circle $S^1 \times 0$ are strictly larger than the ω-limit sets of points starting on $S^1 \times 0$. In the dynamical systems literature such phenomena are referred to as Ω-explosions.

Second, even if convergence to the set of equilibria is established, this does not necessarily imply pointwise convergence to a single equilibrium point. Such pointwise convergence may fail even for smooth gradient systems, as the classical 'Mexican hat' example by Curry [9] shows. This is a gradient system in $\mathbb{R}^2$ with trajectories converging to a full circle of equilibrium points. Thus pointwise convergence to equilibria requires further assumptions, such as normal hyperbolicity of the zero dynamics around the equilibria.

These two aspects, i.e. pointwise convergence to equilibria and the avoiding of Ω-explosions, are addressed by suitable conditions in the subsequent two main results of this paper (for precise definitions see later). The first theorem treats the general case of convergence to compact isolated invariant sets, while the second result establishes pointwise convergence for Morse–Bott zero dynamics.

Theorem 1. *Let $f : M \times \mathbb{R}^m \to TM$ be a smooth family of complete vector fields on a Riemannian manifold M and let $g : \mathbb{R}^m \to \mathbb{R}^m$ be a smooth vector field that has 0 as a globally asymptotically stable equilibrium point. Let $(x, y) \in M \times \mathbb{R}^m$ be any initial point which generates a forward bounded trajectory of (1) and (2). Assume that one of the following two conditions hold.*

(i) *The zero dynamics $\dot{x} = f(x, 0)$ is a gradient flow of a smooth function $\phi : M \to \mathbb{R}$ with compact sublevel sets, such that ϕ is constant on each connected component Λ_i of the set of critical points $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_k$.*
(ii) *M is compact and there exists a Morse decomposition $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_k$ by disjoint, compact invariant sets of the zero dynamics flow on $M \times \{0\}$.*

Then the ω-limit set $\omega(x, y)$ is a chain transitive subset of one of the Morse sets Λ_i.

Thus–under the global asymptotic stability assumption on $\dot{y} = g(y)$– if the zero dynamics are defined by a smooth gradient vector field on M satisfying (i), then the trajectories of the cascade system (1) and (2) converge to the set of equilibrium points on M. Moreover, as is subsequently shown, pointwise convergence holds if the set of equilibria satisfies a normal hyperbolicity condition. We conjecture, that pointwise convergence to an equilibrium point holds, whenever the zero dynamics is a real analytic gradient flow on M.

Theorem 2. *Assume that every forward trajectory of the cascade system (1) and (2) is bounded, with $\dot{y} = g(y)$ globally convergent to a locally exponentially stable equilibrium point $0 \in \mathbb{R}^m$. Assume further that the limiting dynamics $\dot{x} = f(x, 0)$ are Morse–Bott on M. Then every solution $(x(t), y(t))$ of (1) and (2) converges to a single equilibrium point of (3) in $M \times \{0\}$.*

In the above result, the assumed non-cycle condition on the zero dynamics is crucial. The example (6)–(8) shows that there is room for further theoretical improvements, even if the non-cyclicity condition fails, whenever the speed of the zero dynamics is significantly smaller than that of the driving dynamics. We refer to [2] for results in this direction.

2. Convergence to isolated invariant sets

Before stating and proving our main results, we recall some basic facts from the qualitative theory of differential equations. The crucial result used subsequently is the Butler–McGehee Lemma for convergence to isolated invariant sets. In the sequel we assume that M is a complete connected Riemannian manifold; such a Riemannian metric exists on every smooth manifold, but explicit forms of such will not be needed in the subsequent arguments. Let $\mathrm{dist}(x, y)$ denote the minimal length of piecewise smooth curves that connect a point $x \in M$ with $y \in M$. This defines a metric on M, which is complete by the Hopf–Rinow theorem. The associated topology coincides with the manifold topology and therefore any continuous map on M is also continuous with respect to this metric.

Consider a smooth, complete vector field $f : M \to TM$ on a Riemannian manifold M, let $\phi : \mathbb{R} \times M \to M$, $(t, x) \mapsto \phi_t(x)$ denote the associated flow. The ω-limit set of a point $x \in M$ then is the set

$$\omega(x) := \left\{ \lim_{k\to\infty} \phi_{t_k}(x) \mid t_k > 0, t_k \to \infty \right\}. \quad (9)$$

Similarly, the α-limit set is defined by $\alpha(x) := \{\lim_{k\to\infty} \phi_{t_k}(x) \mid t_k < 0, t_k \to -\infty\}$. Detailed information about such limit sets becomes crucial in classical stability theory. For example, the celebrated invariance principle [10] asserts that the omega limit set of any positively bounded trajectory $\{\phi_t(x) \mid t \geq 0\}$ is a nonempty, compact, connected subset of M, that is invariant under the flow. In the sequel, we need a stronger form of the invariance principle that goes back to [11,12]. Let $A \subset M$ denote a nonempty invariant set for the flow of the vector field f on M. Then A is called *internally chain transitive*, if for any $a, b \in A$ and any $\epsilon, T > 0$ there exist finitely many points $x_1, \ldots, x_m \in A$ with $x_1 = a, x_m = b$, and times $T_1, \ldots, T_{m-1} \geq T$ such that $\mathrm{dist}(\phi_{T_i}(x_i), x_{i+1}) < \epsilon$ for $i = 1, \ldots, m - 1$. The sequence $x_1, \ldots, x_m$ then is called an (ϵ, T)-chain in A connecting a with b. The amazing fact then is that limit sets are always internally chain transitive. We formulate the result using Lyapunov functions, although general statements within the abstract context of topological dynamics are possible; see [13,11,12]. Recall, that a smooth function $V : M \to \mathbb{R}$ with compact sublevel sets $\{x \in M \mid V(x) \leq c\}$, $c \in \mathbb{R}$, is called a *Lyapunov-function* of f, if the Lie derivative $L_f V(x) := \mathrm{d}V(x)f(x)$ satisfies $L_f V(x) \leq 0$ for all $x \in M$.

Proposition 3 (*Strong Invariance Principle*). *Let $f : M \to TM$ be a smooth complete vector field on a manifold M and $V : M \to \mathbb{R}$ a Lyapunov function with compact sublevel sets. Let A denote the closed, maximal invariant subset of $\{x \in M \mid L_f V(x) = 0\}$. Then A is compact and the omega limit set $\omega(x)$ of any point $x \in M$ is a nonempty, compact, connected, invariant subset of A that is internally chain-transitive.*