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Cascade representation and minimal factorization of multidimensional systems

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ABSTRACT

We give necessary and sufficient conditions for both the existence of minimal factorization of a rational function of N complex variables and the cascade representation of a Fornasini–Marchesini linear system. In particular, we generalize the well known result of Bart, Gohberg, Kaashoek and Van Doren on minimal factorization of a rational function of one complex variable.

1. Preliminaries

1.1. Introduction

Fornasini–Marchesini linear systems, introduced in 1976 in [4], are defined by sets of linear difference equations of the form

$$\begin{aligned} X(t_1,\ldots,t_N) &= A_1X(t_1-1,\ldots,t_N)+\cdots \\ &+A_NX(t_1,\ldots,t_N-1)+B_1U(t_1-1,\ldots,t_N) \\ &+\cdots+B_NU(t_1,\ldots,t_N-1) \qquad (1.1) \\ Y(t_1,\ldots,t_N) &= CX(t_1,\ldots,t_N)+DU(t_1,\ldots,t_N). \qquad (1.2) \end{aligned}$$

Here $U=\{U_\alpha\}_{\alpha\in\mathbb{N}^N}$ is the input signal, $Y=\{Y_\alpha\}_{\alpha\in\mathbb{N}^N}$ is the output signal and $X=\{X_\alpha\}_{\alpha\in\mathbb{N}^N}$ is the state of the system, and the various matrices are of appropriate sizes. We denote the linear system (1.1) as

$$\mathcal{L}=\begin{bmatrix}\mathbb{A} & \mathbb{B}\\ C & D\end{bmatrix},\quad \text{where } \mathbb{A}=\{A_1,A_2,\ldots,A_N\}, \text{ and } \mathbb{B}=\{B_1,B_2,\ldots,B_N\}.$$

We denote by $z=(z_1,\ldots,z_N)$. When the initial state is equal to 0, we have

$$Y(z)=R(z)U(z),$$

where $U(z)$ and $Y(z)$ are the z-transforms of the input and output respectively, and where

$$R(z)=D+C\left(I_n-\sum_{j=1}^N z_jA_j\right)^{-1}\sum_{j=1}^N z_jB_j \qquad (1.3)$$

is the transfer function of the system. In this expression, n is called the dimension of the state space. We denote the realization (1.3) by

$$R(z)=[\mathbb{A},\mathbb{B},C,D](z).$$

We showed in [1] that any matrix-valued rational function $R(z)$ analytic in a neighborhood of the origin can be realized as the transfer function of a system (1.1), that is, of the form $R(z)=[\mathbb{A},\mathbb{B},C,D](z)$ for a suitable choice of $\{A_j\}_{j=1}^N$, $\{B_j\}_{j=1}^N$, C and D.

In the present work, we will restrict ourselves to the case where the input and the output share the same space (and hence the transfer function is square) and the transfer function is invertible at the origin. Unless stated otherwise, we will always assume $D=I_p$. We discuss two closely related topics: factorization of the transfer function and cascade representation of the linear system. In particular, we generalize the following well known theorem due to Bart, Gohberg, Kaashoek and Van Dooren (see [3]), determining when a rational function of one complex variable admits a minimal factorization:

Theorem 1.1. *Let $R(z)=I_p+C(I_n-zA)^{-1}zB$ be a minimal realization of the transfer function $R(z)$. Then $R(z)$ admits a minimal factorization if and only there exists a decomposition $\mathbb{C}^n=\mathcal{U}\oplus\mathcal{V}$ such that $\mathcal{U}$ is A invariant and $\mathcal{V}$ is $A^\times=A-BC$ invariant.*

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The paper is organized as follows: we devote the rest of this section to giving some more definitions and background material. In Section 2 we study the problem of minimal factorization, and in Section 3 we present the main result of the paper from the linear system point of view.

1.2. Minimal realization and similarity

Two Fornasini–Marchesini systems

$$\mathcal{L}_1 = \begin{bmatrix} \mathbb{A}_1 & \mathbb{B}_1 \\ C_1 & D_1 \end{bmatrix} \quad \text{and} \quad \mathcal{L}_2 = \begin{bmatrix} \mathbb{A}_2 & \mathbb{B}_2 \\ C_2 & D_2 \end{bmatrix}$$

are said to be similar if $D_1 = D_2$ and if there exists an invertible matrix S such that

$$A_{1,j} = S^{-1}A_{2,j}S, \qquad C_1 = C_2 S, \qquad B_{1,j} = S^{-1}B_{2,j},$$

for $j = 1, 2, \ldots, N$.

Definition 1.2. Let $R(z)$ be a rational function analytic in a neighborhood of the origin. A realization $R(z) = [\mathbb{A}, \mathbb{B}, C, D](z)$ is said to be a *minimal realization* if for every other realization $R(z) = [\mathbb{A}_1, \mathbb{B}_1, C_1, D_1](z)$ the dimension of the matrices $A_{1,j}$ is larger or equal to the dimension of the A_j. If $R(z) = [\mathbb{A}, \mathbb{B}, C, D](z)$ is a minimal realization, the dimension of the A_j is called the *FM-degree* of the function $R(z)$.

We use the notation of FM-degree rather than degree for the following reason: When realizing a transfer function of N complex variables, one may consider more than one structure; in return, different realizations will correspond to different types of linear systems (such as the Roesser model and its corresponding realization). In consequence, different realizations may lead to different notions of minimality.

Since in the present study we only refer to the Fornasini–Marchesini model, we will omit the fact that the degree of a realization is the FM-degree – still it should be understood that whenever the notations of minimality and degree are used – the intent is that of FM-degree.

We will denote the FM-degree of $R(z)$ by $\deg(R(z))$. It is a well known fact that while in the one dimensional case, all minimal realizations of a given transfer function are similar, this is no longer the case for N-dimensional systems. In the setting of one dimensional systems, it is also well known that the degree of a transfer function (as defined above) coincides with the McMillan degree of the function- the number of singular points of $R(z)$ (counting multiplicity and ∞). In the multidimensional setting there is no such connection. In fact, it is not clear at all how to define the McMillan degree of a multi-variable function, as a multi-variable polynomial is – in general – not a product of linear terms.

Still, from a system theoretic point of view, the degree of a function $R(z)$ can manifest itself in a fairly simple manner: Assume the input output relation is given via the transfer function. By the main theorem of [1], this input–output relation can be realized through a Fornasini–Marchesini linear system. Clearly, many (in fact, infinitely many) systems may be considered. A natural question is then: what is the minimal possible dimension of the state space needed to realize the input–output relation defined by $R(z)$? It is a trivial observation that the answer to the last question is exactly the FM-degree of $R(z)$.

1.3. Multiplication and inversion of transfer functions

Let $R_1(z) = [\mathbb{A}_1, \mathbb{B}_1, C_1, D_1](z)$ and $R_2(z) = [\mathbb{A}_2, \mathbb{B}_2, C_2, D_2](z)$ be two rational functions, such that $R_1(z)R_2(z)$ is well defined. It is well known (and can be easily obtained by direct calculations) that the product $R(z) = R_1(z)R_2(z)$ can be written as $R(z) = [\mathbb{A}, \mathbb{B}, C, D](z)$ where ($j = 1, 2, \ldots, N$)

$$\begin{aligned} A_j &= \begin{pmatrix} A_{1,j} & B_{1,j}C_2 \\ 0 & A_{2,j} \end{pmatrix}, \qquad B_j = \begin{pmatrix} B_{1,j}D_2 \\ B_{2,j} \end{pmatrix}, \\ C &= \begin{pmatrix} C_1 & D_1C_2 \end{pmatrix}, \qquad D = D_1D_2. \end{aligned} \tag{1.4}$$

(See, for instance, [2, p. 6], [5, Chapter 7] for the one variable case, the multi-variable case is a trivial generalization.)

From these formulas it is clear that for every rational functions $R_1(z)$ and $R_2(z)$ it holds that

$$\deg(R_1(z)R_2(z)) \le \deg(R_1(z)) + \deg(R_2(z)). \tag{1.5}$$

However, in the general case, the above realization need not be minimal, hence it is not clear when (and if) equality is met in (1.5). This issue is studied in Section 2.

Next, assume that $R(z)$ is $\mathbb{C}^{p\times p}$-valued rational function such that $R(0) = I_p$. Denoting

$$A_j^{\times} = A_j - B_jC, \quad j = 1, 2, \ldots, N, \tag{1.6}$$

one can show through direct calculation that $R^{-1}(z)$ has the following realization:

$$R^{-1}(z) = I_p - C\left(I - \sum_{j=1}^{N} z_jA_j^{\times}\right)^{-1} \sum_{j=1}^{N} z_jB_j.$$

In the present study, the above formula for the inverse will not be needed, but expressions (1.6) play a key role (see Theorems 2.2 and 3.2).

1.4. Cascades of linear systems

Consider the case where we are given two different Fornasini–Marchesini systems

$$\mathcal{L}_1 = \begin{bmatrix} \mathbb{A}_1 & \mathbb{B}_1 \\ C_1 & D_1 \end{bmatrix} \quad \text{and} \quad \mathcal{L}_2 = \begin{bmatrix} \mathbb{A}_2 & \mathbb{B}_2 \\ C_2 & D_2 \end{bmatrix},$$

with transfer functions $R_1(z)$ and $R_2(z)$ respectively. Assume that we wish to use the output of the second system as the input of the first one. The input output relation of the new system can be written as:

$$\mathcal{L} = \begin{bmatrix} \left\{\begin{pmatrix} A_{1,j} & B_{1,j}C_2 \\ 0 & A_{2,j} \end{pmatrix}\right\}_{j=1}^{N} & \left\{\begin{pmatrix} B_{1,j}D_2 \\ B_{2,j} \end{pmatrix}\right\}_{j=1}^{N} \\ \begin{pmatrix} C_1 & D_1C_2 \end{pmatrix} & D_1D_2 \end{bmatrix}. \tag{1.7}$$

(See [5] p. 270 for the one dimensional case, generalization to N-dimensional systems is straightforward.) We will refer to the system $\mathcal{L}$ as the *cascade* product of the two systems $\mathcal{L}_1$ and $\mathcal{L}_2$, and will denote it by $\mathcal{L}_1 \circ \mathcal{L}_2$.

With $\mathcal{L} = \begin{bmatrix} \mathbb{A} & \mathbb{B} \\ C & D \end{bmatrix}$, by formula (1.4), it is easy to see that $R(z) = [\mathbb{A}, \mathbb{B}, C, D](z)$ is a realization for the rational matrix valued function $R_1(z)R_2(z)$.

2. Minimal factorization of a transfer function

As stated before, the input–output relation can be given completely via the transfer function. In this section, we provide the main contribution of this study from the transfer function point of view (Theorem 2.2). We start with the following definition:

Definition 2.1. Let $R(z)$ be a $\mathbb{C}^{p\times p}$-valued rational function. The factorization $R(z) = R_1(z)R_2(z)$ (here $R_j : \mathbb{C}^N \to \mathbb{C}^{p\times p}, \quad j = 1, 2$) is *minimal* if $\deg(R(z)) = \deg(R_1(z)) + \deg(R_2(z))$.