Systems & Control Letters 58 (2009) 342–345

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Copositive Lyapunov functions for switched systems over cones

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ARTICLE INFO

Article history:
Received 23 August 2007
Received in revised form
19 December 2008
Accepted 19 December 2008
Available online 4 February 2009

Keywords:
Copositive Lyapunov functions
Stability theory

ABSTRACT

We answer two open questions on copositive Lyapunov functions which were recently posed by M.K. Çamlıbel and J.M. Schumacher in the book *Unsolved Problems in Mathematical Systems and Control Theory*, edited by V.D. Blondel and A. Megretski [M.K. Çamlibel, J.M. Schumacher, Copositive Lyapunov functions, in: V.D. Blondel, A. Megretski (Eds.), Unsolved Problems in Mathematical Systems and Control Theory, Princeton University Press, 2004, pp. 189–193. Available online at http://press.princeton.edu/math/blondel/]. These questions are: what are necessary and sufficient conditions for the existence of a Lyapunov function for a linear system which is defined over a cone? How can this be extended to switched linear systems where the system matrix varies over time?

We present conditions answering these questions. Our conditions amount to checking feasibility or infeasibility of a system of linear inequalities.

1. Introduction

Consider a linear system of the form $\dot{x} = Ax$ with the initial value condition $x(0) = x_0$. It is well known that this system is asymptotically stable if and only if a quadratic Lyapunov function $x^T Px$ exists, i.e., if and only if there exists a symmetric matrix P such that the Lyapunov inequality is fulfilled, i.e.,

$$P \succ 0 \quad \text{and} \quad A^{\mathrm{T}}P + PA \prec 0, \tag{1}$$

where the notation $M \succ 0$ ($M \prec 0$) indicates that M is positive (negative) definite.

In some applications, one is interested in so called positive systems, i.e., systems for which $x(0) \geq 0$ (componentwise) implies $x(t) \geq 0$ for all $t \geq 0$, cf. [2]. More generally, one may be interested in a system

$$\dot{x} = Ax \quad \text{for } Cx \geq 0 \tag{2}$$

for a given matrix $C \in \mathbb{R}^{m\times n}$. In this setting, stability of the system is related to existence of a matrix P such that the definiteness conditions of (1) hold with respect to the cone $\mathcal{C} = \{x \in \mathbb{R}^n : Cx \geq 0\}$ rather than with respect to the whole space $\mathbb{R}^n$.

Semidefiniteness with respect to a cone $\mathcal{C}$ is also called copositivity with respect to $\mathcal{C}$. We say that a matrix M is copositive w.r.t. $\mathcal{C}$ (or $\mathcal{C}$-copositive, denoted by $M \overset{\mathcal{C}}{\succeq} 0$) if $x \in \mathcal{C}$ implies $x^{\mathrm{T}}Mx \geq 0$. If equality only holds for $x = 0$, then M is strictly $\mathcal{C}$-copositive ($M \overset{\mathcal{C}}{\succ} 0$). A matrix which is copositive w.r.t. $\mathbb{R}^n_+$ is simply called copositive. $\mathcal{C}$-copositivity is a generalization of both copositivity (choose $\mathcal{C} = \mathbb{R}^n_+$) and positive semidefiniteness (choose $\mathcal{C} = \mathbb{R}^n$). For a good introduction to this topic see [3].

When discussing stability of system (2), $\mathcal{C}$-copositivity arises naturally when we ask for generalizations of the Lyapunov inequality (1). This is the first problem posed by Çamlıbel and Schumacher [1]:

Problem 1: Let a square matrix A and a cone $\mathcal{C} = \{x \in \mathbb{R}^n : Cx \geq 0\}$ be given. Determine necessary and sufficient conditions for the existence of a symmetric matrix P such that $P \overset{\mathcal{C}}{\succ} 0$ and $A^{\mathrm{T}}P + PA \overset{\mathcal{C}}{\prec} 0$.

In contrast to positive definiteness, copositivity of a matrix cannot be checked through its eigenvalues. In fact, checking whether a given matrix is copositive is a co-NP-complete problem (see [4]), i.e., no polynomial algorithm for this is known. Several conditions for copositivity of a matrix have been developed (cf. [5,6]), most of which rely on properties of principal submatrices. Spectral properties of copositive matrices have been studied in [7]. $\mathcal{C}$-copositivity with respect to polyhedral cones has been dealt with in [8–10].

None of these approaches seems appropriate in our context because all of them propose conditions for a given fixed matrix to be copositive. Our problem is a different one: given a system matrix A and a cone $\mathcal{C}$, decide whether or not there exists a $\mathcal{C}$-copositive matrix with $A^{\mathrm{T}}P + PA \overset{\mathcal{C}}{\prec} 0$. Geometrically, this problem can be formulated as: decide whether the two matrix cones $\{P \in \mathbb{R}^{n\times n} : P \overset{\mathcal{C}}{\succ} 0\}$ and $\{P \in \mathbb{R}^{n\times n} : A^{\mathrm{T}}P + PA \overset{\mathcal{C}}{\prec} 0\}$ intersect.

As a second step of generalization, one may want to consider a system where the system matrix A and the cone $\mathcal{C}$ are not fixed,

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0167-6911/$ – see front matter
doi:10.1016/j.sysconle.2008.12.006

but switch between matrices A_1, A_2 and cones $\mathcal{C}_1, \mathcal{C}_2$, respectively, over time. This means, one considers a system of the form

$$\dot{x} = A_i x \quad \text{for} \quad C_i x \geq 0, \quad i = 1, 2. \tag{3}$$

Finding conditions that guarantee asymptotic stability of a piecewise linear system $\dot{x} = A_i x$ with A_i in some family of matrices has attracted a considerable amount of attention, cf. [11, 12] and references therein, whereas switched systems over cones have not yet been studied thoroughly. Problem 2 of Çamlıbel and Schumacher [1] is related to this question:

Problem 2: Let two square matrices A_1, A_2 and two cones $\mathcal{C}_1 = \{x \in \mathbb{R}^n : C_1 x \geq 0\}$ and $\mathcal{C}_2 = \{x \in \mathbb{R}^n : C_2 x \geq 0\}$ be given. Determine sufficient conditions for the existence of a symmetric matrix P such that $P \overset{\mathcal{C}_i}{\succ} 0$ and $A_i^T P + PA_i \overset{\mathcal{C}_i}{\prec} 0$ for $i = 1, 2$.

For the case $n = 2$, necessary and sufficient conditions, which can be efficiently checked, are presented in [13]. It is also shown that these do not hold in dimensions higher than 2. In [14], necessary and sufficient criteria for the existence of common linear copositive Lyapunov functions of positive systems (i.e., the cone considered is $\mathcal{C} = \mathbb{R}^n_+$ throughout) for arbitrary n are proven. Unfortunately, to check those conditions one needs to investigate 2^n matrices and check whether or not they are Hurwitz, which limits this approach to small dimensions.

In the case $\mathcal{C} = \mathbb{R}^n_+$ and $n \leq 4$ it is known (see [15]) that the cone of copositive matrices equals the Minkowski-sum of the cone of positive semidefinite matrices and the cone of entrywise nonnegative matrices. Therefore, in this situation the copositive cone can be described by a finite system of linear matrix inequalities (LMIs), and thus the existence of P can be efficiently checked using semidefinite programming techniques. In higher dimensions, it is an open question whether there exists a finite system of LMIs describing the copositive cone.

In this paper, we propose a method to answer the questions of Problems 1 and 2 in general dimensions. Our approach does not characterize classes of matrices A_i for which the answer is positive. Instead, we focus on deciding algorithmically for given matrices A_i and cones $\mathcal{C}_i$ whether or not there exists a matrix P with the desired properties.

In Section 2, we discuss copositivity of a matrix with respect to a cone $\mathcal{C}$ and provide necessary and sufficient conditions for a matrix to be $\mathcal{C}$-copositive. In Section 3, we show how these conditions apply to solving Problems 1 and 2, and we illustrate the approach by an example.

2. Conditions for cone-copositivity

In this section, we show how techniques developed in [16] for copositive matrices can be applied to $\mathcal{C}$-copositivity. We derive sufficient conditions and show that these conditions eventually capture all strictly $\mathcal{C}$-copositive matrices.

$\mathcal{C}$-copositivity of a matrix means that the induced quadratic form is nonnegative for all $x \in \mathcal{C}$. A first observation is that it suffices to consider points of unit norm:

Lemma 1. *Let $\|\cdot\|$ denote any norm on $\mathbb{R}^n$. We have*

$$M \overset{\mathcal{C}}{\succeq} 0 \iff x^T M x \geq 0 \quad \text{for all } x \in \mathcal{C} \text{ with } \|x\| = 1.$$

An analogous result holds for $M \overset{\mathcal{C}}{\succ} 0$.

In our context, it turns out to be advantageous to use the 1-norm. We will use the notation

$$\mathcal{B}_1 := \{x \in \mathbb{R}^n : \|x\|_1 = 1\}$$

to denote its unit sphere. The last lemma establishes that, when formulating conditions for $\mathcal{C}$-copositivity of a matrix, it is sufficient to consider points in $\mathcal{C} \cap \mathcal{B}_1$.

Note that $\mathcal{B}_1 \subset \mathbb{R}^n$ is a union of $(n-1)$-dimensional simplices (i.e., a union of sets each of which is the convex hull of n affinely independent points). The next lemma gives an easily verifiable sufficient condition for nonnegativity of a quadratic form over a simplex.

Lemma 2. *Let Δ be a simplex. Denote by V the set of its vertices and by E the set of its edges. If*

$$\begin{aligned} &v^T M v \geq 0 \ \ (\text{resp. } v^T M v > 0) \quad \text{for all } v \in V, \text{ and} \\ &u^T M v \geq 0 \ \ (\text{resp. } u^T M v > 0) \quad \text{for all } (u, v) \in E, \end{aligned} \tag{4}$$

then $x^T M x \geq 0$ (resp. $x^T M x > 0$) for all $x \in \Delta$.

Proof. Let $V = \{v_1, \ldots, v_n\}$. We can represent each point x in the affine hull of Δ by its uniquely defined barycentric coordinates $\lambda = \lambda(x) = (\lambda_1, \ldots, \lambda_n)$ with respect to Δ:

$$x = \sum_{i=1}^n \lambda_i v_i \quad \text{with} \quad \sum_{i=1}^n \lambda_i = 1.$$

With this representation, we get

$$x^T M x = \left(\sum_{i=1}^n \lambda_i v_i\right)^T M \left(\sum_{j=1}^n \lambda_j v_j\right) = \sum_{i,j=1}^n v_i^T M v_j \lambda_i \lambda_j.$$

For $x \in \Delta$, we have $\lambda(x) \geq 0$, whence (4) implies $x^T M x \geq 0$. □

In order to derive conditions for $\mathcal{C}$-copositivity of a matrix, we partition the relevant set $\mathcal{C} \cap \mathcal{B}_1$ into simplices and apply Lemma 2 to each of those. Observe that the cone $\mathcal{C} = \{x \in \mathbb{R}^n : Cx \geq 0\}$ we are concerned with is a polyhedron, and that $\mathcal{B}_1 = \{x \in \mathbb{R}^n : \|x\|_1 = 1\}$ is a union of $(n-1)$-dimensional polytopes. Therefore, the intersection $\mathcal{C} \cap \mathcal{B}_1$ of both is also a union of polytopes, and it is not hard to find a simplicial partition of $\mathcal{C} \cap \mathcal{B}_1$. By this we mean the following:

Definition 3. Let Ω be any set in $\mathbb{R}^n$. A family $\mathcal{P} = \{\Delta^1, \ldots, \Delta^m\}$ of simplices satisfying

$$\Omega = \bigcup_{i=1}^m \Delta^i \qquad \text{and} \qquad \operatorname{int} \Delta^i \cap \operatorname{int} \Delta^j = \emptyset \ \text{ for } \ i \neq j$$

is called a *simplicial partition* of Ω.

For convenience, we denote by $V_{\mathcal{P}}$ the set of all vertices of simplices in $\mathcal{P}$, and by $E_{\mathcal{P}}$ the set of all edges of simplices in $\mathcal{P}$.

Note that the set $E_{\mathcal{P}}$ is not the same as $V_{\mathcal{P}} \times V_{\mathcal{P}}$, because a partition $\mathcal{P}$ may contain more than one simplex, and for vertices u and v from different simplices, $(u, v) \notin E_{\mathcal{P}}$.

If we are dealing with positive systems $\dot{x} = Ax$, i.e., if $\mathcal{C} = \mathbb{R}^n_+$, then we can start with $\mathcal{P} = \{\Delta\}$, where $\Delta = \{x \in \mathbb{R}^n_+ : \|x\| = 1\}$. Otherwise, we need to determine the intersection of $\mathcal{C} \cap \mathcal{B}_1$ and find a simplicial partition thereof.

If the extremal rays of the cone $\mathcal{C}$ are known it is possible to transform the cone-copositivity to ordinary copositivity using Corollary 2.21 from [17].

Given a simplicial partition of $\mathcal{C} \cap \mathcal{B}_1$, the next theorem gives a sufficient condition for the matrix to be copositive:

Theorem 4. *Let M be a symmetric matrix. Let $\mathcal{P}$ be a simplicial partition of $\mathcal{C} \cap \mathcal{B}_1$.*

(a) *If $v^T M v \geq 0$ for all $v \in V_{\mathcal{P}}$ and $u^T M v \geq 0$ for all $(u, v) \in E_{\mathcal{P}}$, then M is $\mathcal{C}$-copositive.*
(b) *If $v^T M v > 0$ for all $v \in V_{\mathcal{P}}$ and $u^T M v > 0$ for all $(u, v) \in E_{\mathcal{P}}$, then M is strictly $\mathcal{C}$-copositive.*