Systems & Control Letters 58 (2009) 276–281

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

A remark on abstract multiplier conditions for robustness problems

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ARTICLE INFO

Article history:
Received 15 April 2008
Received in revised form
10 November 2008
Accepted 12 November 2008
Available online 17 December 2008

Keywords:
Multipliers
Full-block *S*-procedure
Implicit uncertain system
Linear matrix inequalities
Uncertain systems

ABSTRACT

This paper presents a generalization of recent results of Açıkmeşe and Corless [B. Açıkmeşe, M. Corless, Stability analysis with quadratic Lyapunov functions: Some necessary and sufficient multiplier conditions, Systems Control Letters 57 (2008) 78–94] concerning multiplier conditions of quadratic stability of uncertain/nonlinear systems. Abstract full block S-procedure results are formulated, extending the results of Scherer [C.W. Scherer, LPV control and full block multipliers, Automatica 37 (2001) 361–375] for such cases when the sets representing the uncertainties do not necessarily have subspace structure. The main contribution of the present work is the investigation of the conditions under which the results of Scherer and Açıkmeşe and Corless can be formulated in a unified framework.

1. Introduction

Robust analysis and synthesis problems for linear systems depending on uncertain time-varying parameters have attracted considerable attention. Since several nonlinear systems can be interpreted within this framework, the importance of the results goes far beyond the linear systems. Different techniques have been proposed to tackle such problems; one of them is based on Lyapunov methods. In this line, the introduction of the concept of quadratic stability has been one of the first important contributions, and it has proven to be very useful, both in analysis and synthesis of uncertain/nonlinear control systems (see e.g. [1,2] and the references therein).

Recently, Açıkmeşe and Corless [2] have proven some necessary and sufficient conditions for quadratic stability involving multiplier matrices. The system they investigated can be described as

$$\dot{x} = Ax + Bp, \quad q = Cx + Dp, \quad (q, p) \in \Omega, \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the state and $p(t) \in \mathbf{R}^{l_p}$ is the collection of all time-dependent uncertain/nonlinear elements in the system, $q(t) \in \mathbf{R}^{l_q}$, A, B, C, D are known matrices of appropriate size and Ω is some known subset of $\mathbf{R}^{l_q} \times \mathbf{R}^{l_p}$ characterizing the time-dependent nonlinear and uncertain connections. Several classes of uncertain/nonlinear terms, important in practical applications, are presented and analyzed in the paper. It should be noted that no a priori algebraic or topological requirements on Ω are imposed.

In [2], a system described by (1) is called quadratically stable with decay rate $\alpha > 0$ and Lyapunov matrix $P = P^{\mathrm{T}} > 0$ if

$$\begin{pmatrix} x \\ p \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} PA + A^{\mathrm{T}}P + 2\alpha P & PB \\ B^{\mathrm{T}}P & 0 \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix} \le 0 \tag{2}$$

for all x and p which satisfy

$$(Cx + Dp, p) \in \Omega. \tag{3}$$

In consequence of (3), the satisfaction of (2) is very hard to analyze, and it is also difficult to find appropriate α and P. In order to partially decouple (2) and (3) the authors of [2] propose to introduce certain multiplier matrices. In [2], a symmetric matrix M is called a multiplier matrix for Ω if

$$\begin{pmatrix} q \\ p \end{pmatrix}^{\mathrm{T}} M \begin{pmatrix} q \\ p \end{pmatrix} \ge 0 \quad \text{for all } (q, p) \in \Omega. \tag{4}$$

Under some technical assumptions, it has also been proven in [2] that system (1) is quadratically stable with a decay rate $\overline{\alpha}$ and Lyapunov matrix P if, and only if, for each $\alpha \in (0, \overline{\alpha})$ there is a matrix M in a sufficiently rich family of multipliers (for a definition, see [2] and Section 2) such that

$$\begin{pmatrix} PA + A^{\mathrm{T}}P + 2\alpha P & PB \\ B^{\mathrm{T}}P & 0 \end{pmatrix} + \begin{pmatrix} C & D \\ 0 & I \end{pmatrix}^{\mathrm{T}} M \begin{pmatrix} C & D \\ 0 & I \end{pmatrix} \le 0. \tag{5}$$

Under some additional requirements on Ω, an analogous result is formulated for strict inequalities of the form (2) and (5). In the

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doi:10.1016/j.sysconle.2008.11.004

proofs, the specific structure of (2) and (5) is highly exploited. We emphasize the importance of the idea of a 'sufficiently rich family of multipliers': this gives the possibility of considering a *subset* of multipliers not losing the equivalence of inequalities like (2), (3) and (5). The authors of [2] analyze, in detail, the appropriate families of multipliers for a very rich collection of different types of uncertainties establishing the real decoupling of inequalities (4) and (5).

The idea of the (full-block) multiplier matrices has appeared much earlier (see e.g. [3] and the references therein) and it has proven to be very useful in many cases: for example, when external perturbations are present, and explicit performance specification is imposed, as well as in guaranteed cost control and game problems, both for continuous and discrete-time systems (to mention but a few applications, see e.g. [4–6,3,7,8]). In these results, the uncertainty is represented by a family of *subspaces* parametrized by a compact set of parameters. The flexibility of this approach is largely due to the abstract formulation of the full-block S-procedure given in [7].

The main purpose of this paper is to generalize the above mentioned results in a unified framework and to analyze what kind of conditions can be formulated for the uncertainty/nonlinearity constraining set which do not restrict the generality.

We emphasize that the present paper does not aim to show such uncertainty sets for which its results are applicable while those of [2] fail. The present paper can be considered as a generalization of the full block S-procedure of [7] and the multiplier method of [2] in the following sense. On the one hand, these results are applicable for a broad class of robust analysis and control design problems (similar to the results of [7]). On the other hand, these results take advantage of considering a general uncertainty/nonlinearity set (similarly to [2]).

The results are summarized in Theorems 1 and 2 which are formulated analogously to Theorems 1 and 2 of [2]. The proofs follow Scherer's thoughts given in [7] with necessary modifications.

Standard notation is applied. Relation $P > 0$ (≥ 0) denotes the positive (semi-) definiteness of a symmetric matrix P, and I_n is the identity matrix of dimension n. The ith real eigenvalue of the symmetric matrix P is denoted by $\lambda_i(P)$, while $\lambda_{\max}(P)$ stands for its maximum eigenvalue. The Moore–Penrose pseudo-inverse of P is denoted by P^{+}, and the transpose of P is P^{T}. Set K is called a cone, if $\lambda x \in K$ for all $x \in K$ and $\lambda > 0$ (see e.g. [9]). The closure of the set Ω is denoted by $\overline{\Omega}$, while the cone generated by Ω is cone $\Omega = \left\{q \in \mathbf{R}^l : q = \mu q_0,\ q_0 \in \Omega,\ \mu > 0\right\}$. (In contrast to [9], this notation is used not only for convex cones.) If Λ is a linear subspace, its orthogonal complement is $\Lambda^{\perp}$. Symbol ker V stands for the kernel of matrix V. The surface of the ball of radius r is denoted by S_r in $\mathbf{R}^N$.

2. Abstract multiplier method

2.1. Nonstrict inequalities

Suppose that $\mathcal{B} \subset \mathbf{R}^N$ is a subspace and matrices $U \in \mathbf{R}^{j\times N}$ and $V \in \mathbf{R}^{l\times N}$ are fixed, where V has maximum row rank. For any given set $\Omega \subset \mathbf{R}^l$ let us define the set $\mathcal{B}_\Omega \subset \mathcal{B}$ as

$$\mathcal{B}_\Omega = \{y \in \mathcal{B} : Vy \in \Omega\}. \tag{6}$$

Let Θ be a closed set of parameters and let $\mathcal{Q}$ be a set-valued map from Θ to the subsets of $\mathbf{R}^l$:

$$\theta \in \Theta \to \mathcal{Q}(\theta) \subset \mathbf{R}^l.$$

Assume that $V\mathcal{B} \cap \mathcal{Q}(\theta) \neq \emptyset$ for all $\theta \in \Theta$.

Let the set $\mathcal{B}_{\mathcal{Q}(\theta)}$ be defined for any $\theta \in \Theta$, according to (6).

Proposition 1. *For any set $\Omega \subset \mathbf{R}^l$,*

$$\operatorname{cone} \mathcal{B}_\Omega = \mathcal{B}_{\operatorname{cone}\Omega}, \tag{7}$$

$$\ker V \cap \mathcal{B} \subset \overline{\operatorname{cone} \mathcal{B}_\Omega}, \tag{8}$$

$$\overline{\operatorname{cone}\mathcal{B}_\Omega} \subset \mathcal{B}_{\overline{\operatorname{cone}\Omega}}. \tag{9}$$

Proof. Relation (7) is a straightforward consequence of the definitions. To show (8) consider a $y \in \ker V \cap \mathcal{B}$. Let $\overline{y} \in \mathcal{B}_\Omega$, $\mu_n > 0$, $\mu_n \to 0$, if $n \to \infty$, and consider $\widetilde{y}_n = \frac{1}{\mu_n} y + \overline{y} \in \mathcal{B}$. Then $V\widetilde{y}_n = \frac{1}{\mu_n} Vy + V\overline{y} = V\overline{y} = \overline{q} \in \Omega$, and $\mu_n \overline{q} \in \operatorname{cone}\Omega$. Hence

$$V(\mu_n \widetilde{y}_n) = Vy + \mu_n V\overline{y} \in \operatorname{cone}\Omega,$$

i.e.

$$\mu_n \widetilde{y}_n \in \mathcal{B}_{\operatorname{cone}\Omega}.$$

Since $\mu_n \widetilde{y}_n = y + \mu_n \overline{y} \to y$, (8) is proven.

It can be seen that $\mathcal{B}_{\overline{\operatorname{cone}\Omega}}$ is closed. Therefore from cone $\Omega \subset \overline{\operatorname{cone}\Omega}$ and from (7), (9) follows immediately. ∎

Consider a symmetric matrix $\Psi \in \mathbf{R}^{j\times j}$. Suppose that we want to investigate the inequality

$$y^{\mathrm{T}} U^{\mathrm{T}} \Psi U y \leq 0, \quad \text{for all } y \in \mathcal{B}_{\mathcal{Q}(\theta)} \quad \text{and for all } \theta \in \Theta. \tag{10}$$

Definition 1 (*[2]*). A symmetric matrix M is called a *multiplier matrix* for $\mathcal{Q}(.)$ if $q^{\mathrm{T}} M q \geq 0$ for all $q \in \mathcal{Q}(\theta)$, for all $\theta \in \Theta$. The set $\mathcal{M}$ of multiplier matrices for $\mathcal{Q}(.)$ is called *sufficiently rich*, if for any $\overline{M}$, which is a multiplier matrix for $\mathcal{Q}(.)$, there exists an element $M \in \mathcal{M}$ such that $M \leq \overline{M}$.

Remark 1. Observe that (10) is satisfied for all $y \in \mathcal{B}_{\mathcal{Q}(\theta)}$ if, and only if, it is satisfied for all $y \in$, cone $\mathcal{B}_{\mathcal{Q}(\theta)}$. Moreover, a symmetric matrix M is a multiplier matrix for $\mathcal{Q}(.)$ if, and only if, the same is true for cone $\mathcal{Q}(\theta)$ for all $\theta \in \Theta$. Therefore, without loss of generality, we may restrict ourself to set-valued mappings, the images of which are cones.

Condition 1. For all $\theta \in \Theta$, $\mathcal{Q}$ is a cone.

Condition 2. For all $\theta \in \Theta$, either $\mathcal{Q}(\theta) \subset V\mathcal{B}$, or any $q \in \mathcal{Q}(\theta)$ can be decomposed as $q = q_1 + q_2$, where $q_1 \in \mathcal{Q}(\theta) \cap V\mathcal{B}$ and $q_2 \in (V\mathcal{B})^{\perp}$.

Remark 2. Observe that, in the investigation of (10), only the set $\mathcal{Q}(\theta) \cap V\mathcal{B}$ plays any role, thus $\mathcal{Q}(\theta) \subset V\mathcal{B}$ might be assumed. In [2], this assumption corresponds to the condition that $\mathbf{R}^{n_q} = \operatorname{Im}(C\ D)$, which is evidently satisfied if matrix C has full row rank. If $\operatorname{Im}(C\ D)$ is a proper subspace of $\mathbf{R}^{n_q}$ then the model describing the uncertainties/nonlinearities can be modified, as proposed in [2], Section 2.4. In [7], $\mathcal{Q}(\theta)$ is a subspace, thus Condition 2 is satisfied as well.

The aim is to get rid of the constraint $y \in \mathcal{B}_{\mathcal{Q}(\theta)}$ at the expense of the introduction of a multiplier matrix.

Theorem 1. *Suppose that Conditions 1 and 2 hold, $U^{\mathrm{T}}\Psi U$ is strictly negative definite on $\ker V \cap \mathcal{B}$, and $\mathcal{M}$ is a sufficiently rich family of multipliers for $\mathcal{Q}(.)$. Then the following statements are equivalent.*

1. *Inequality (10) holds true for all $y \in \mathcal{B}_{\mathcal{Q}(\theta)}$ and $\theta \in \Theta$.*
2. *There exists a multiplier matrix $M \in \mathcal{M}$, which satisfies*

$$y^{\mathrm{T}}\left(U^{\mathrm{T}}\Psi U + V^{\mathrm{T}} M V\right) y \leq 0 \quad \text{for all } y \in \mathcal{B}. \tag{11}$$