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Invariance principles for switching systems via hybrid systems techniques[☆]

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ABSTRACT

Invariance principles and sufficient conditions for asymptotic stability for switching systems are given. Multiple Lyapunov-like functions are used, and dwell-time, persistent dwell-time, and weak dwell-time switching signals are considered. The invariance principles are derived from general invariance principles for hybrid systems. Asymptotic stability is concluded under observability assumptions or common bounds on the Lyapunov-like functions.

1. Introduction

1.1. Background

Switching systems are dynamical systems governed by a differential equation whose right hand side is selected from a given family of functions, based on some (time or state dependent) switching rule. These systems are a particular class of hybrid dynamical systems as they combine continuous dynamics (differential equations) with discrete dynamics (switching). Over the last fifteen years, the area of switching systems has been very active and many efforts have been made to study their stability properties. These include the early work on sufficient conditions for asymptotic stability of linear switching systems with multiple Lyapunov-like functions in [20,19] and of nonlinear switching systems in [12,2,5,15]. Asymptotic stability under particular classes of switching signals has been analyzed in [11,15,9,10,1,18]. For much more background, see [16,15,9].

In this paper, we focus on tools for convergence analysis of solutions to switching systems under certain classes of switching signals. On this topic, [9] introduced an invariance principle for switched linear systems under *persistently dwell-time* switching signals. The follow-up work, [10], extended some of the results of [9] to a family of nonlinear switching systems; [1] presented invariance principles for nonlinear switching systems with *dwell-time* switching signals and *state-dependent* switching that, as a difference to [9], allow for locally Lipschitz Lyapunov functions. The paper [18] stated an invariance principle for nonlinear switching systems with *average dwell-time* signals and underlined the role of "sequential compactness" of particular subsets of solutions to switching systems in invariance arguments. For hybrid systems, [17] extended LaSalle's invariance principle to nonblocking, deterministic, and continuous hybrid systems, while in [4], an invariance principle for left-continuous and impulsive systems without multiple jumps at an instant (and with further quasi-continuity properties including uniqueness of solutions) is presented. More recently, in [21], invariance principles were shown for general hybrid systems in the framework of [8]. (That framework allows for nonuniqueness of solutions, multiple jumps at time instants, and Zeno behaviors, while only posing mild regularity conditions on the data.)

1.2. Contribution

The goal of this paper is to show how some of the results of [21] can be used to obtain invariance principles for switching systems, under various types of switching signals. While doing that, we recover, generalize, and/or strengthen some of the results of [9,10,1,18]. In particular:

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- Corollary 5.3 strengthens [1, Theorems 1,2] by including both forward and backward invariance conditions on the set to which solutions converge. Also, Corollary 4.4, while having the same invariance conditions as those in [1], also incorporates level sets of Lyapunov functions into the description of the invariant set.
- Corollary 4.7 is an invariance principle for nonlinear switching systems that generalizes [9, Theorem 8] stated for linear switching systems. Even in the linear setting, Corollary 4.7 yields smaller, in comparison to [9, Theorem 8], sets to which solutions converge. This is possible thanks to taking into account the period of persistence of solutions.
- [10, Theorem 7] is derived, in Corollary 4.13, as a consequence of the hybrid invariance principle in Theorem 4.1.
- First conclusion of [18, Theorem 2.3] follows from Lemma 4.12 and Corollary 4.13.

In deriving the results, we rely on invariance principles in [21], but only in proving Theorems 4.1 and 5.2. (Then, several consequences of these two theorems are derived in a self-contained way.) We also use two techniques that should prove useful for purposes other than those in this paper. More specifically:

- Given a solution to a switching system, $\tau_D > 0$, and a sequence of time intervals of length at least τ_D on which the logical mode takes on a particular value q^*, one can identify the restriction of the solution to those intervals with a function on $[0, \infty)$. The resulting object is not a solution to a switching system, as the continuous variable of the original switching system may now be only piecewise continuous. (Indeed, for the original system there is no reason for the continuous variable to have the same value at the end of an interval when the mode is q^* and at the beginning of the next interval when the mode is q^* again.) However, this resulting object is a solution to an appropriately formulated hybrid system (truly hybrid system, in which both the "continuous" variable and the logical mode may jump). That hybrid system can be given sufficient regularity properties, like those called for by [8]. Then, invariance principles of [21] can be applied to it, with implications for the original switching system. See the proofs of Corollaries 4.4 and 4.7 for illustrations of this technique.
- In the case of multiple Lyapunov functions, i.e., when in logical mode q, a function V_q is decreasing at a rate W_q, it is often assumed that the value of V_{q^*} at the end of an interval with mode q^* is greater or equal than the value of V_{q^*} at the beginning of the next interval with mode q^*. It follows that the function $(x, q) \mapsto V_q(x)$ cannot be used in the standard Lyapunov sense, as it is not necessarily decreasing along solutions — it can increase during switches between different logical modes. However, it can be shown that for each bounded solution (x, q) to the switching system, the function $(x, q) \mapsto W_q(x)$ is integrable. (A similar technique was used, for example, in [10, Theorem 7].) This paves way to the application of invariance principles of [21] that rely on an output function that decreases sufficiently fast to 0. See the proof of Theorem 5.2 for an illustration.

In presenting the results, we clearly separate the statements only about invariance of sets to which bounded solutions of switched systems converge (Corollaries 4.4, 4.7 and 5.3) from stronger statements about asymptotic stability that rely on additional information like observability or common bounds on Lyapunov functions (Corollaries 4.10 and 4.13).

Some results of this paper were previously announced in [7].

2. Preliminaries

2.1. Switching systems

Let $O \subset \mathbb{R}^n$ be an open set, let $Q := \{1, 2, \ldots, q_{\max}\}$, and for each $q \in Q$, let $f_q : O \to \mathbb{R}^n$ be a continuous function. We consider switching systems given by

$$\mathcal{SW} : \dot{x} = f_q(x). \tag{1}$$

For more background on switching systems, see e.g. [15,9].

A complete *solution to the switching system* $\mathcal{SW}$ consists of a locally absolutely continuous function $x : [0, \infty) \to O$ and a function $q : [0, \infty) \to Q$ that is piecewise constant and has a finite number of discontinuities in each compact time interval, and $\dot{x}(t) = f_{q(t)}(x(t))$ for almost all $t \in [0, \infty)$. We will say that a complete solution (x, q) to $\mathcal{SW}$ is precompact if x is bounded with respect to O, that is, there exists a compact set $K \subset O$ such that $x(t) \in K$ for all $t \in [0, \infty)$.

In this paper, we will consider only complete solutions to $\mathcal{SW}$ that are generated under particular classes of switching signals. Let (x, q) be a complete solution to $\mathcal{SW}$ and let $t_0 = 0$, and $t_1, t_2, \ldots$ be the consecutive (positive) times at which q is discontinuous (informally, t_i is the time of the i-th switch). The solution (x, q) is a *dwell-time solution* with dwell time $\tau_D > 0$ if $t_{i+1} - t_i \geq \tau_D$ for $i = 0, 1, \ldots$. (That is, jumps are separated by at least τ_D amount of time.) The solution (x, q) is a *persistent dwell-time solution* with persistent dwell time $\tau_D > 0$ and period of persistence $T > 0$ if there exists a subsequence $0 = t_{i_0}, t_{i_1}, t_{i_2}, \ldots$ of the sequence $\{t_i\}$ such that $t_{i_k+1} - t_{i_k} \geq \tau_D$ for $k = 1, 2, \ldots$ and $t_{i_{k+1}} - t_{i_k+1} \leq T$ for $k = 0, 1, \ldots$. (That is, at most T amount of time passes between two consecutive intervals of length at least τ_D on which there are no jumps.) Finally, a solution (x, q) is a *weak dwell-time solution* with dwell time $\tau_D > 0$ if there exists a subsequence $0 = t_{i_0}, t_{i_1}, t_{i_2}, \ldots$ of the sequence $\{t_i\}$ such that $t_{i_k+1} - t_{i_k} \geq \tau_D$ for $k = 1, 2, \ldots$. (That is, there are infinitely many intervals of length τ_D with no switching.) These classes of solutions follow the definitions in [9], see also [11]. More precisely, in [9], dwell-time solutions to $\mathcal{SW}$ are elements of the set $\mathcal{S}_{dwell}$, persistent dwell-time solutions to $\mathcal{SW}$ are elements of the set $\mathcal{S}_{p\text{-}dwell}$, and weak dwell-time solutions to $\mathcal{SW}$ are elements of $\mathcal{S}_{weak\text{-}dwell}$.

2.2. Hybrid systems

We consider hybrid systems of the form

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D \end{cases} \tag{2}$$

with an associated state space $\mathcal{O} \subset \mathbb{R}^m$. Above, F (respectively, G) is the possibly set-valued map describing the flow, (respectively, the jumps) while C (respectively, D) is the set on which the flow can occur (respectively, from which the jumps can occur). For more background on hybrid systems in this framework, see [6] or [8].

A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid time domain* if $E = \bigcup_{j=0}^{J-1} \left([t_j, t_{j+1}], j\right)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \cdots \leq t_J$. It is a *hybrid time domain* if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, \ldots, J\})$ is a compact hybrid time domain. Equivalently, E is a hybrid time domain if E is a union of a finite or infinite sequence of intervals $[t_j, t_{j+1}] \times \{j\}$, with the "last" interval possibly of the form $[t_j, T)$ with T finite or $T = \infty$. A *hybrid arc* is a function whose domain is a hybrid time domain (for a hybrid arc x, its domain will be denoted dom x) and such that for each $j \in \mathbb{N}$, $t \to x(t, j)$ is locally absolutely continuous on dom $x \cap ([0, \infty) \times \{j\})$.

A hybrid arc x is a *solution to the hybrid system* $\mathcal{H}$ if $x(0, 0) \in C \cup D$, $x(t, j) \in \mathcal{O}$ for all $(t, j) \in$ dom x, and