Available online at www.sciencedirect.com
ScienceDirect
Systems & Control Letters 56 (2007) 342–348
SYSTEMS & CONTROL LETTERS
www.elsevier.com/locate/sysconle

Solution of the input-constrained LQR problem using dynamic programming

José B. Mare*, José A. De Doná

Centre for Complex Dynamic Systems and Control, School of Electrical Engineering and Computer Science, The University of Newcastle, Callaghan 2308, NSW, Australia

Received 3 June 2004; received in revised form 12 February 2005; accepted 7 October 2006
Available online 28 November 2006

Abstract

The input-constrained LQR problem is addressed in this paper; i.e., the problem of finding the optimal control law for a linear system such that a quadratic cost functional is minimised over a horizon of length *N* subject to the satisfaction of input constraints. A *global* solution (i.e., valid in the *entire* state space) for this problem, and for arbitrary horizon *N*, is derived *analytically* by using dynamic programming. The scalar input case is considered in this paper. Solutions to this problem (and to more general problems: state constraints, multiple inputs) have been reported recently in the literature, for example, approaches that use the geometric structure of the underlying quadratic programming problem and approaches that use multi-parametric quadratic programming techniques. The solution by dynamic programming proposed in the present paper coincides with the ones obtained by the aforementioned approaches. However, being derived using a different approach that exploits the *dynamic* nature of the constrained optimisation problem to obtain an *analytical* solution, the present result complements the previous methods and reveals additional insights into the intrinsic structure of the optimal solution.

Keywords: Constrained optimal control; Input constraints; Dynamic programming; Analytical solution

1. Introduction

The solution of constrained LQR problems has attracted considerable attention recently. This interest is, mainly, due to the fact that these problems constitute the core underlying optimisation problem that is solved, at each sampling time, by model predictive control algorithms (one of the most popular control methodologies used in industry at present). Of particular interest has been the derivation of *explicit* solutions that, through a characterisation of the optimal solution that is computed *off-line*, would render on-line optimisation unnecessary.

Recently, two approaches have simultaneously been developed, aiming at obtaining such off-line explicit solutions. These two approaches have been reported in, for example, [2,11]. The first method provides an algorithm, based on multi-parametric quadratic programming techniques, to obtain an explicit solution to the problem. The second method mentioned above uses geometric arguments to obtain a characterisation of the optimal solution of the resulting quadratic programme. Subsequently, many interesting extensions have been reported. For example, in [9] an explicit solution to a Min–Max MPC problem with bounded uncertainties is obtained; a suboptimal formulation that reduces the complexity of the solution is proposed in [7]; in [6] the infinite-time solution is computed by combining multi-parametric quadratic programming with reachability analysis. In fact, there exists a growing number of publications on this topic that reflects the interest that these problems have generated.

Starting from a different perspective to the ones mentioned above, a solution to the input-constrained case has been reported in [4]. This solution, obtained by using dynamic programming arguments, was of a *local* nature; i.e., valid in a region of the state space, and consisted in simply *clipping* the optimal unconstrained solution; i.e., $u = -\mathrm{sat}_{\Delta}(Kx)$, where $\mathrm{sat}_{\Delta}(\cdot)$ is the saturation function with bounds $\pm\Delta$. (Related work that utilises a different approach based on KKT optimality conditions has also been published in [8]). In [3], the region where this solution is valid was further characterised by a set of linear inequalities and it was shown that, inside this region, the controller $u = -\mathrm{sat}_{\Delta}(Kx)$ effectively reaches the constraints, thus providing a nontrivial characterisation of

* Corresponding author.
E-mail addresses: jose.mare@studentmail.newcastle.edu.au (J.B. Mare), jose.dedona@newcastle.edu.au (J.A. De Doná).

0167-6911/$ - see front matter
doi:10.1016/j.sysconle.2006.10.018

the optimal solution. These results were used in [5] to obtain improved terminal constraint sets that guarantee closed-loop stability of model predictive control schemes.

In the present paper, the solution by dynamic programming is further extended to provide the *global* solution (i.e., valid in the *entire* state space) to the problem for arbitrary horizon *N*. The global solution is, of course, not just $u=-\mathrm{sat}_\Delta(Kx)$ and can be concisely summarised by the expression $u=-\mathrm{sat}_\Delta(\hat{L}_{N,i}x+\hat{h}_{N,i})$ for $x \in X_i$, where the set $X_i \subset \mathbb{R}^n$ represents a region of a state-space partition $\mathbb{R}^n=\bigcup_j X_j$. The solution presented in this paper exploits the *dynamic* nature of the optimisation problem to obtain an analytical characterisation of the optimal control law. Although the final result (when evaluated at any given particular problem) obviously coincides, by optimality, with those obtained by other methods, the main contribution of the solution obtained by dynamic programming lies in that all the derivations required are *analytical* and, hence, the solution is given by closed-form expressions (i.e., all the expressions are closed-form functions of the data of the problem). One of the motivations of this approach is that it provides an alternative methodology, based on analytical derivations, to obtain the solution. It is envisaged that this alternative methodology could be amenable to being extended to related open problems, such as constrained control of non-linear systems, closed-form solution of the constrained continuous-time LQR problem, etc.

The remainder of the paper is organised as follows. In Section 2, the input-constrained LQR problem is formulated. In Section 3 the solution is provided, which comprises the control law structure and the regions of the state space where each component of the control law is valid. The derivation of the solution is done by dynamic programming and is included in the Appendix at the end of the paper. The solution is illustrated with an example in Section 4. Finally, Section 5 presents the conclusions.

2. Problem formulation

Consider the discrete-time linear state-space model

$$x_{k+1}=Ax_k+Bu_k, \tag{1}$$

where $x_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}$ are the state and control input, respectively. In (1) the pair (A,B) is assumed to be stabilisable, and the control input is required to satisfy the constraint

$$u_k \in \Omega,$$

where $\Omega \triangleq [-\Delta,\Delta]$, $\Delta>0$. The following notation will be employed. The *control sequence*, for some horizon *N*, is denoted by

$$\mathbf{u} \triangleq \mathbf{u}_0 \triangleq \{u_0,u_1,\ldots,u_{N-1}\}.$$

For $r \in \{1,\ldots,N\}$, and for some initial time $N-r \in \{0,1,\ldots,N-1\}$, let $\mathbf{u}_{N-r}$ denote the *partial control sequence*:

$$\mathbf{u}_{N-r} \triangleq \{u_{N-r},u_{N-(r-1)},\ldots,u_{N-1}\}.$$

By $\mathbf{u} \in \Omega^N$ ($\mathbf{u}_{N-r} \in \Omega^r$) we denote the case in which each element in the sequence satisfies $u_k \in \Omega$, $k=0,\ldots,N-1$ ($k=N-r,\ldots,N-1$).

The solution of (1) at time $k \geqslant N-r$ when the initial state at time $N-r$ is $x_{N-r}=x$, and the control sequence is $\mathbf{u}_{N-r}$, is denoted by $x_k^{\mathbf{u}_{N-r}}(x,N-r)$. To simplify notation, the initial time is dropped when it is zero; i.e., $x_k^{\mathbf{u}}(x) \triangleq x_k^{\mathbf{u}_0}(x,0)$. The fixed-horizon optimal control problem considered is

$$\mathscr{P}_N(x): V_N^{\mathrm{o}}(x)=\min_{\mathbf{u}} V_N(x,\mathbf{u})$$
$$\text{subject to } \mathbf{u} \in \Omega^N. \tag{2}$$

The cost $V_N(\cdot,\cdot)$ in (2) is defined by

$$V_N(x,\mathbf{u})=\sum_{k=0}^{N-1}(x_k^{\mathrm{T}}Qx_k+u_k^{\mathrm{T}}Ru_k)+x_N^{\mathrm{T}}Px_N, \tag{3}$$

with $x_k=x_k^{\mathbf{u}}(x)$, and where Q is the state weighting matrix, assumed to be positive semidefinite, R is the control weighting matrix, assumed to be positive definite, and P is the terminal state weighting matrix which is chosen as the positive definite matrix solution of the algebraic Riccati equation

$$P=A^{\mathrm{T}}PA+Q-K^{\mathrm{T}}\bar{R}K, \tag{4}$$

where

$$K \triangleq \bar{R}^{-1}B^{\mathrm{T}}PA, \quad \bar{R} \triangleq R+B^{\mathrm{T}}PB. \tag{5}$$

It is well known (see, for example, [10]) that with this choice of *terminal weight P*, and provided that the horizon *N* is large enough, the resulting *receding horizon* implementation of the control law gives an asymptotically stable closed loop system and possesses all the properties of infinite-horizon optimal control. By the receding horizon implementation it is understood the standard technique (also known as model predictive control) in which the first control action u_0 in the optimal control sequence $\mathbf{u}$ that minimises (2)–(3) is applied to system (1) and, as the state evolves to a new value in the next sampling time, the optimisation process is repeated over a horizon of length *N* (receding horizon).

3. Solution of $\mathscr{P}_N(x)$ by dynamic programming

For each $r \in \{1,\ldots,N\}$, the *partial value function* (or *optimal cost to go*) is defined by

$$V_r^{\mathrm{o}}(x)=\min_{\mathbf{u}_{N-r}} V_r(x,\mathbf{u}_{N-r}) \tag{6}$$

subject to the constraint $\mathbf{u}_{N-r} \in \Omega^r$, where the *partial cost* $V_r(\cdot,\cdot)$ is defined by

$$V_r(x,\mathbf{u}_{N-r})=\sum_{k=N-r}^{N-1}(x_k^{\mathrm{T}}Qx_k+u_k^{\mathrm{T}}Ru_k)+x_N^{\mathrm{T}}Px_N \tag{7}$$