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Systems & Control Letters 56 (2007) 349–356
SYSTEMS & CONTROL LETTERS
www.elsevier.com/locate/sysconle

A power-based description of standard mechanical systems

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Received 7 June 2004; received in revised form 28 April 2006; accepted 10 October 2006
Available online 4 December 2006

Abstract

This paper is concerned with the construction of a power-based modeling framework for mechanical systems. Mathematically, this is formalized by proving that every standard mechanical system (with or without dissipation) can be written as a gradient vector field with respect to an indefinite metric. The form and existence of the corresponding potential function is shown to be the mechanical analog of Brayton and Moser's mixed-potential function as originally derived for nonlinear electrical networks in the early sixties. In this way, several recently proposed analysis and control methods that use the mixed-potential function as a starting point can also be applied to mechanical systems.

Keywords: Mechanical systems; Hamiltonian mechanics; Brayton–Moser equations

1. Introduction and motivation

It is well known that a large class of physical systems (e.g., mechanical, electrical, electromechanical, thermodynamical, etc.) admits, at least partially, a representation by the Euler–Lagrange (EL) or Hamiltonian equations of motion, see e.g. [1,7,9,13,14], and the references therein. A key aspect for both sets of equations is that the energy storage in the system plays a central role. For standard mechanical systems with n degrees of freedom, and locally represented by n generalized displacement coordinates $q = \mathrm{col}(q_1, \ldots, q_n) \in \mathbb{Q}$, the EL equations of motion are given by[1]

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\nabla_{\dot{q}}\mathscr{L}(q,\dot{q})\right) - \nabla_q\mathscr{L}(q,\dot{q}) = \tau, \tag{1}$$

where $\dot{q} = \mathrm{col}(\dot{q}_1, \ldots, \dot{q}_n) \in \mathbb{V}$ denotes the generalized velocities, and $\mathscr{L} : \mathbb{Q} \times \mathbb{V} \to \mathbb{R}$ represents the Lagrangian which is defined by the difference between the kinetic co-energy and the potential energy. Usually the forces τ are decomposed into dissipative forces and generalized external forces.

The relation between the EL equations and the Hamiltonian equations is classically established as follows. Defining the generalized momenta $p = \nabla_{\dot{q}}\mathscr{L}(q,\dot{q})$ with $p = \mathrm{col}(p_1, \ldots, p_n) \in \mathbb{P}$, the equations of motion, as described by the set of second-order equations (1), can be written as a set of $2n$ first-order equations:

$$\dot{q} = \nabla_p\mathscr{H}(q,p), \quad \dot{p} = -\nabla_q\mathscr{H}(q,p) + \tau. \tag{2}$$

Here, $\mathscr{H} : \mathbb{Q} \times \mathbb{P} \to \mathbb{R}$ denotes the Hamiltonian which represents the sum of the kinetic and potential energy.

The relationship between (1) and (2) is graphically represented in the diagram shown in Fig. 1 (solid lines). Clearly, the diagram suggests that there exists a dual form of (1) in the sense that a mechanical system can be expressed in terms of a set of generalized momenta and its time derivatives, which represent a set of generalized forces. Indeed, in [7] a description of the dynamics in the generalized momentum and force spaces $\mathbb{P}$ and $\mathbb{F}$, respectively, is called a co-Lagrangian system, where the Lagrangian $\mathscr{L}$ in (1) is replaced by its dual form $\mathscr{L}^* : \mathbb{P} \times \mathbb{F} \to \mathbb{R}$, representing the difference between the potential co-energy and the kinetic energy, while the forces τ are replaced by external velocities τ^*, i.e.,

$$\frac{\mathrm{d}}{\mathrm{dt}}(\nabla_{\dot{p}}\mathscr{L}^*(p,\dot{p})) - \nabla_p\mathscr{L}^*(p,\dot{p}) = \tau^*, \tag{3}$$

with $\dot{p} = \mathrm{col}(\dot{p}_1, \ldots, \dot{p}_n) \in \mathbb{F}$. Hence, the co-Lagrangian system (3) represents a velocity-balance equation.

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[1] By ∇_x we denote the partial derivative operator $\partial/\partial x$.

0167-6911/$ - see front matter
doi:10.1016/j.sysconle.2006.10.015

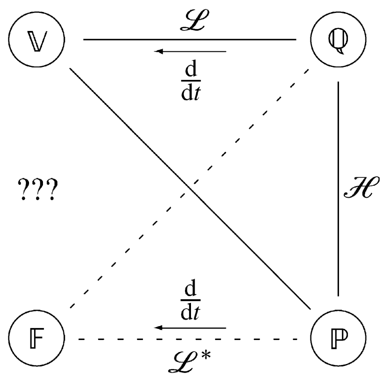

Fig. 1. Mechanical configuration space quadrangle: the symbols $\mathbb{Q}$, $\mathbb{P}$, $\mathbb{V}$ and $\mathbb{F}$ denote the spaces of the generalized displacements, momenta, velocities and forces. The solid and dashed diagonal lines represent the directions for the Legendre transformations of the Lagrangian and co-Lagrangian, respectively, in relation to the Hamiltonian; the question marks denote the fourth equation set to be explored in this paper. Notice that the relation between the spaces $\mathbb{Q}$ and $\mathbb{V}$, and similarly between $\mathbb{P}$ and $\mathbb{F}$, is the $\mathrm{d}/\mathrm{d}t$ operator.

So far we have considered three possible representations describing the dynamics of a standard mechanical system. The underlying relationship between the three sets of equations is the existence of the Legendre transformations between $\mathbb{Q}$, $\mathbb{V}$, $\mathbb{P}$ and $\mathbb{F}$. Furthermore, the quadrangle of Fig. 1 also suggests a fourth equation set. Intuitively, at this point, one could be tempted to call a dynamic description on the spaces $\mathbb{V}$ and $\mathbb{F}$ the *co-Hamiltonian* equations of motion. Starting from the Hamiltonian equation set, if both the Legendre transformations of $\mathbb{Q} \to \mathbb{F}$ and $\mathbb{P} \to \mathbb{V}$ are considered simultaneously, one obtains $\mathscr{H}^* : \mathbb{F} \times \mathbb{V} \to \mathbb{R}$ which appears to be a bona-fide co-Hamiltonian candidate. Hence, based on the latter observation, and in comparison to (2), this would suggest that the 'co-Hamiltonian' equation set should read

$$\dot{v} = \nabla_f \mathscr{H}^*(v, f), \quad \dot{f} = -\nabla_v \mathscr{H}^*(v, f) + \phi,$$

where $v = \dot{q}$, and $f = \dot{p}$. However, the latter set of equations is *not* correctly describing the dynamics. Furthermore, it is not clear how the external signals, represented by ϕ, relate to the original external signals τ and/or τ^*. Thus, the existence, form and meaning of the fourth description remains to be clarified.

In this paper, it is our objective to identify the fourth equation set (indicated by the question marks) in the quadrangle of Fig. 1, and to formally complete the relationships between the different sets of equations. It will be shown that the fourth equation set constitutes a mechanical analog of the Brayton–Moser (BM) equations [2]. These equations represent a gradient system with respect to an indefinite metric defined by the dynamic part of the network (capacitors and inductors), and a mixed-potential function which describes the static part (interconnection, resistors and sources) of the network and has the units of power. Besides the completion of the quadrangle, the mechanical analog of the BM equations can be useful for other features like:

- Stability analysis along the lines of [2]. The mixed-potential (and thus the power in the system) can be used to construct Lyapunov-type functions to prove stability under certain conditions—even in cases that the system contains (regions of) negative resistance! Additionally, the stability criteria stemming from this method can be used to find lower bounds on the control parameters when applying passivity-based control (PBC),[2] see e.g., [5] for some recent results in the field of electronic power converter control.
- Definition of new passivity properties along the lines of [3]. This includes the definition of alternative conjugated port-variables (inputs and outputs) with respect to an alternative storage function (i.e., the mixed-potential).
- The notion of the aforementioned new passivity properties have led to the paradigm of power-shaping stabilization. Some recent applications to nonlinear RLC circuits have been reported in [8]. The power-shaping method is based on a particular selection of the input signals (the controls) as to shape the power flow (read: the mixed-potential).
- The BM equations seem to be a natural equation set in relation with bond-graph theory since the state variables live in the flow and effort spaces.

There exists a widely accepted standard analogy between simple mechanical and electrical system elements, like the 'spring–capacitor' and the 'mass–inductor' analogy used in this paper, but also the 'spring-inductor' and 'mass-capacitor' analogy used in e.g. [11]. However, the existence of a well-defined analogy for more general mechanical systems is not straightforward. One of the main reasons for making such analogy difficult is the presence of the coriolis and centrifugal forces, which do not appear as such in the electrical domain. Another difficulty is that, in contrast to electrical networks, mechanical systems are in general not nodical. Hence, a mechanical system cannot always be considered as an interconnected graph. For these reasons, we can, in general, not equate the dynamics of a mechanical system mutatis mutandis along the lines of [2]. A more dedicated analysis is needed and a dedicated transformation algorithm that goes beyond the Legendre transformation needs to be developed.

Although there have been earlier attempts towards the formalization of a mechanical analog of [2], see e.g., [4,6], in our opinion, the mechanical analog of [2] presented in this paper seems a rather natural and general one. The approach of our paper differs from [10] in the sense that here we consider a description starting from the Hamiltonian system equations, and possibly staying within the original generalized position and generalized momenta coordinates. In [10] the starting point is given by the EL equations and an electrical interpretation in canonical BM coordinates of e.g., the gravity force as well as the coriolis and centrifugal terms is given. Also, the final BM form is different.

The structure of the paper is as follows. Section 2 discusses the original form of the BM equations. In Section 3, a lemma

[2] PBC is a control method that has its roots in the field of robotics and the closely related Lagrangian framework. For a detailed elaboration on this subject the interested reader is referred to [9], and the references cited therein.