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Hopf bifurcation control: A new approach

Fernando Verduzco[a,*], Joaquin Alvarez[b]

[a] Sonora University (UNISON), Mexico
[b] Scientific Research and Advanced Studies Center of Ensenada (CICESE), Mexico

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Abstract

In this paper we find conditions to control a Hopf bifurcation in a class of nonlinear systems whose linear approximation has two eigenvalues on the imaginary axis, without assuming that the system is controllable. We use the center manifold theorem to project the dynamics on a two dimensional manifold, and design a controller that permits us to decide the stability and direction of the emerging periodic solution.

Keywords: Hopf bifurcation; Center manifold; Bifurcation control

1. Introduction

In the last years there has been a great interest to analyze control systems displaying complex dynamics. An emerging research field that has become very stimulating is the bifurcation control which, among other objectives, aims to modify the dynamical behavior of a system around bifurcation points, generates a particular bifurcation at a given parameter value [3], delays the onset of an inherent bifurcation [10], or stabilizes a bifurcated solution [1,2]. An overview of this field can be found in [5].

In [1,2,11,12] the bifurcation control problem is analyzed using state feedback. Kang and others [4,7,8] investigated this problem using normal forms and invariants. For the particular case of Hopf bifurcation, several procedures have been proposed to design the controller. In [1] the stability of the emerging periodic orbit is determined by its characteristic exponents. The center manifold theorem is used in [7] to project the system dynamics on a two dimensional manifold. They decompose the original system in two subsystems, a two-dimensional, non-hyperbolic system and a $(n-2)$-dimensional, hyperbolic one; then they represent the hyperbolic part in the Brunovski form, assuming controllability of this subsystem. Verduzco and others [11,12] analyzed the control of Hopf bifurcation in a similar way as Hamzi and others [7], removing the controllability assumption by using a Jordan form representation for the hyperbolic subsystem. In this document we follow the last method and complement the results obtained in [11].

2. Preliminary results

Two essential theorems are included in this section. The center manifold theorem reduces the analysis of the system to dimension two, while the Hopf bifurcation theorem establishes conditions to ensure the direction and stability of the limit cycle.

Theorem 1 (*The local Center Manifold Theorem, Perko [9]*). *Let $f \in \mathscr{C}^r(\mathscr{U})$, where $\mathscr{U}$ is an open subset of $\mathbb{R}^n$ containing the origin and $r \geqslant 1$. Suppose that $f(0)=0$ and that $Df(0)$ has c eigenvalues with zero real parts and s eigenvalues with negative real*

* Corresponding author. Tel.: +52 662 2592155; fax: +52 662 2592219.
E-mail address: verduzco@gauss.mat.uson.mx (F. Verduzco).

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parts, where $c+s=n$. Then the system $\dot{\xi}=f(\xi)$ can be written in diagonal form

$$\dot{x}=Ax+F_1(x,y),\ \dot{y}=By+F_2(x,y),$$

where $(x,y)\in\mathbb{R}^c\times\mathbb{R}^s$, $A\in\mathbb{R}^{c\times c}$ has c eigenvalues with zero real parts, $B\in\mathbb{R}^{s\times s}$ has s eigenvalues with negative real parts, $F_i(0)=0$ and $DF_i(0)=0$, for $i=1,2$. Furthermore, there exists a c-dimensional invariant center manifold $W^c_{\mathrm{loc}}(0)$ tangent to the center eigenspace E^c at 0, given by

$$W^c_{\mathrm{loc}}(0)=\{(x,y)\in\mathbb{R}^c\times\mathbb{R}^s\,|\,y=h(x)\ \text{for}\ x\in N_\delta(0)\},$$

where $N_\delta(0)$ is a neighborhood of the origin, with radius δ, and $h\in\mathscr{C}^r(N_\delta(0))$ satisfies

$$Dh(x)(Ax+F_1(x,h(x)))-Bh(x)-F_2(x,h(x))=0.$$

Finally, the flow on the center manifold is defined by the system of differential equations

$$\dot{x}=Ax+F_1(x,h(x)),$$

for $x\in N_\delta(0)$.

Theorem 2 (*Hopf Bifurcation Theorem, Guckenheimer and Holmes [6]*). *Suppose that the system $\dot{x}=f(x,\mu)$, $x\in\mathbb{R}^n$, $\mu\in\mathbb{R}$, has an equilibrium point (x_0,μ_0) such that*

(A1) *$D_xf(x_0,\mu_0)$ has a simple pair of pure imaginary eigenvalues and no other eigenvalues with zero real parts.*
(A2) *Let $\lambda(\mu)$, $\bar{\lambda}(\mu)$ be the eigenvalues of $D_xf(x_0,\mu_0)$ which are imaginary at $\mu=\mu_0$, such that*

$$\frac{\mathrm{d}}{\mathrm{d}\mu}(\mathrm{Re}(\lambda(\mu)))|_{\mu=\mu_0}=d\neq 0. \tag{1}$$

Then there is a unique three-dimensional center manifold passing through $(x_0,\mu_0)\in\mathbb{R}^n\times\mathbb{R}$ and a smooth system of coordinates for which the Taylor expansion of degree three on the center manifold, in polar coordinates, is given by

$$\dot{r}=(d\mu+ar^2)r,\ \dot{\theta}=\omega+c\mu+br^2.$$

If $a\neq 0$, then there is a surface of periodic solutions in the center manifold which has quadratic tangency with the eigenspace of $\lambda(\mu_0)$, $\bar{\lambda}(\mu_0)$ agreeing to second order with the paraboloid $\mu=-ar^2/d$. If $a<0$, then these periodic solutions are stable, while if $a>0$, they are repelling limit cycles.

There exists an expression for bidimensional systems to find the *first Lyapunov coefficient a* [6]. Consider the system

$$\dot{x}=Jx+F(x),$$

where $J=\begin{pmatrix}0 & -\omega\\ \omega & 0\end{pmatrix}$, $F(x)=\begin{pmatrix}F_1(x)\\ F_2(x)\end{pmatrix}$, $F(0)=0$, and $DF(0)=0$. Then

$$a=\frac{1}{16\omega}(R_1+\omega R_2), \tag{2}$$

where

$$R_1=F_{1x_1x_2}(F_{1x_1x_1}+F_{1x_2x_2})-F_{2x_1x_2}(F_{2x_1x_1}+F_{2x_2x_2})-F_{1x_1x_1}F_{2x_1x_1}+F_{1x_2x_2}F_{2x_2x_2},$$
$$R_2=F_{1x_1x_1x_1}+F_{1x_1x_2x_2}+F_{2x_1x_1x_2}+F_{2x_2x_2x_2}.$$

There exists another way to express R_2. If $F(x)=\frac{1}{2}\mathscr{Q}(x,x)+\frac{1}{6}\mathscr{C}(x,x,x)+\cdots$, where $\mathscr{Q}=\begin{pmatrix}Q_1\\ Q_2\end{pmatrix}$, $\mathscr{C}=\begin{pmatrix}C_{11} & C_{21}\\ C_{21} & C_{22}\end{pmatrix}$ with $Q_i, C_{ij}\in\mathbb{R}^{2\times 2}$, then

$$R_2=\mathrm{tr}(\mathscr{C})=\mathrm{tr}(C_{11}+C_{22}), \tag{3}$$

with $\mathrm{tr}(\cdot)=\mathrm{trace}(\cdot)$.