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Pre-control form of the generalized Campbell–Baker–Hausdorff–Dynkin formula for affine nonholonomic systems

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Abstract

In this paper a pre-control form of the generalized Campbell–Baker–Hausdorff–Dynkin (gCBHD) formula is presented. This form is dedicated to nonholonomic (affine) systems often encountered in robotics. The simplest possible expression for the pre-control gCBHD is obtained as control-dependent functions pre-multiply elements of the Ph. Hall basis. Algorithmic aspects of deriving the formula are highlighted. An application of the pre-control form of the gCBHD formula in motion planning of nonholonomic systems is also reported.

Keywords: Generalized Campbell–Baker–Hausdorff–Dynkin formula; Affine systems; Nonholonomic systems; Lie algebra; Motion planning

0. Nomenclature

P_r denotes a set of all permutations of the set: $\{1, \ldots, r\}$, e.g. $P_3 = \{(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)\}$.

$e(\sigma \in P_r)$ is the number of errors in the permutation: $\sigma = \{\sigma(1), \sigma(2), \ldots, \sigma(r)\}$, with the error increased by one each time when the next element in the permutation is smaller than the current one. E.g. $e((1, 2, 3)) = 0$, $e((1, 3, 2)) = 1$, $e((3, 2, 1)) = 2$.

$T_r(t)$ is the r dimensional simplex, $T_r(t) = \{s \in R^r : 0 < s_1 < s_2 < \cdots < s_r < t\}$. Related issues: integration over the simplex $\int_{T_r(t)} = \int_{s_r=0}^{t} \int_{s_{r-1}=0}^{s_r} \cdots \int_{s_1=0}^{s_2}$, and description of the integration variables $\mathrm{d}s^r = \mathrm{d}s_1 \mathrm{d}s_2 \ldots \mathrm{d}s_r$.

$[X, Y]$ is a Lie bracket of X, Y vector fields, in coordinates $[X, Y] = (\partial Y/\partial q)X - (\partial X/\partial q)Y$ where q is the vector of coordinate variables;

Generators $Y = \{Y_1, \ldots, Y_r\}$ are the vector fields spanning a free Lie algebra $\mathrm{L}A(Y)$.

Monomials are vector fields arising by Lie bracketing of generators, e.g. $[[Y_2, Y_1], Y_2] \in \mathrm{Mon}$, $Y_1 + [Y_1, Y_2] \notin \mathrm{Mon}$.

Degree is a function assigning to a monomial its degree according to the formula:

$$\mathrm{degree}(Y_i) = 1, \quad i = 1, \ldots, r,$$

$$\mathrm{degree}([X_1, X_2]) = \mathrm{degree}(X_1) + \mathrm{degree}(X_2). \tag{1}$$

E.g. $\mathrm{degree}([[Y_3, Y_1], Y_2]) = 3$, $\mathrm{degree}(Y_2) = 1$.

Layer is a subset of monomials with constant degree:

$$L_i = \{X \in \mathrm{Mon} \mid \mathrm{degree}(X) = i\} \tag{2}$$

$[Y_1, Y_2] \in L_2$, $Y_2 \in L_1$.

Index of a monomial is a vector whose components show a number of times consecutive generators appear in a given vector field:
$\mathrm{index}(X) = (\#Y_1(X), \#Y_2(X), \ldots, \#Y_r(X))$, where $\#Y(X)$ denotes a number of occurrences of the generator Y in the vector field X. E.g. for $r = 3$: $\mathrm{index}([Y_1, [Y_1, Y_2]]) = (2, 1, 0)$, $\mathrm{index}(Y_3) = (0, 0, 1)$. It is easy to see that for the Lie monomial X its degree equals $\mathrm{degree}(X) = \sum_{i=1}^{r} \#Y_i(X)$. A partial ordering of indices can be introduced. Let two

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monomials X, Y be given

$$\begin{aligned}\mathrm{index}(X) < \mathrm{index}(Y) \\ \Leftrightarrow \forall_i\, \mathrm{index}(X)_i \leqslant \mathrm{index}(Y)_i \wedge \exists_i\ \mathrm{index}(X)_i \\ < \mathrm{index}(Y)_i,\end{aligned} \tag{3}$$

where $\mathrm{index}(Y)_i$ denotes the ith component of the vector $\mathrm{index}(Y)$.

Operators are the following Lie operators acting on a compound Lie monomial:

Op1 antisymmetry: $[B, A] \to -[A, B]$,
Op2 Jacobi identity: $[A, [B, C]] \to \{-[C, [A, B]], [B, [A, C]]\}$.

The properties are presented, in rather unorthodox fashion, as production rules rather than as equalities. Production rules are used in symbolic programming languages (like LISP) to (transform) simplify Lie monomials. Later on components of a sum will be presented as lists (sets). This notation is due to implementation issues where each element of a sum is processed separately and its selection is easier from the list rather than from the additive form.

Monomials, as can be seen from the Jacobi identity, are not independent each of other. An independent collection of monomials forms a basis.

H Ph. Hall basis is a set of monomial $H = \{H_1, H_2, H_3, H_4, \ldots\}$ which form a basis for the free Lie algebra LA. There can be generated many equivalent Ph. Hall bases. In sequel a natural Ph. Hall basis will be preferred where items in each layer are ordered with respect to the degree of its most left (sub-)arguments. When there are two items with the same structure (degrees of each respective sub-monomials of compared monomials are the same) and the aforementioned condition is useless, the lexicographic ordering is used to decide their ordering. The effective algorithm to generate this basis is presented in [5]. Two examples explain details of the ordering: $[Y, [X, [X, [X, Y]]]] \overset{\mathrm{H}}{<} [[X, Y], [X, [X, Y]]]$-degrees decide, $[X, [X, Y]] \overset{\mathrm{H}}{<} [Y, [X, Y]]$-lexicographic ordering decides. For two generators, X, Y, the very first items of the Ph. Hall basis are the following: $X, Y, [X, Y], [X, [X, Y]], [Y, [X, Y]], [X, [X, [X, Y]]], [Y, [X, [X, Y]]], [Y, [Y, [X, Y]]], \ldots$.

1. Introduction

The generalized Campbell–Baker–Hausdorff–Dynkin (gCBHD) formula describes (locally) a solution to a nonautonomous system of differential equations with a given initial condition

$$\dot{q}(t) = A(t)(q(t)), \quad q(0) = q_0, \tag{4}$$

where $A(t)(\cdot)$ is a family of analytic vector fields parameterized continuously by t. A solution has the form

$$q(t) = \exp z(t)\,(q(0)) \simeq z(t)(q(0)) + q(0), \tag{5}$$

where $\exp z(t)(q_0)$ is a solution of $(\mathrm{d}/\mathrm{d}s)v(s,t) = z(t)v(s)$ with the boundary condition $v(0,0) = q_0$. $v(s,t) = \exp(s\,z(t))(q_0) \Rightarrow q(t) = v(1,t) = \exp z(t)(q_0)$. For $t \to 0$, $z(t)$ takes a form of the series [22]:

$$z(t) \sim \sum_{r=1}^{\infty} \sum_{\sigma \in P_r} \frac{(-1)^{e(\sigma)}}{r^2 \binom{r-1}{e(\sigma)}} \cdot \int_{T_r(t)} [[\ldots[A(s_{\sigma(1)}), A(s_{\sigma(2)})]\ldots], A(s_{\sigma(r)})]\,\mathrm{d}s^r. \tag{6}$$

The gCBHD formula is a transport operator, cf. (5) which, for small t, moves a current state q_0 of a system (4) in the direction $z(t)(q_0)$.

In nonholonomic robotics (mobile robotics, free-floating vehicles, underwater vessels, underactuated manipulators) special subclasses of the systems (4) are of interest. Generally, they can be described by the equations

$$\dot{q}(t) = A(t)(q) = \sum_{i=1}^{m} g_i(q) u_i, \tag{7}$$

where $\dim q > \dim u$, g_i are vector fields called generators, u denotes controls. When nonholonomic systems are considered at a kinematic level, the controls u_i in (7) are principally unconstrained and the system (7) is without a drift. When dynamics are also taken into account a drift is present and the system (7) is affine one. The drift can be modeled by setting $u_1 \equiv 1$. Later on systems (7) with and without a drift will be considered simultaneously.

In the papers [7,21] simplifications of the gCBHD formula (5,6) were obtained for general, nonautonomous systems (4). Simplifications were aimed at the reduction of the number of items in the Lie series (6) (without any simplifications the number of Lie monomials in (6), for fixed r is equal to $r!$). In this paper, being extended version of [9], simplified version of the gCBHD formula for more specific, nonholonomic systems (7) will be presented. Applications of the gCBHD formula in nonholonomic motion planning are numerous [6,8]. Roughly speaking, the gCBHD formula allows to express, locally around a given point in the state space $Q \ni q$, admissible directions of motions parameterized with parameters of controls. Consequently, the motion in a desired direction can be reformulated as the optimization task of determining parameters of controls that steer the nonholonomic system into the desired direction.

As the gCBHD formula is valid only locally (in a neighborhood of a given state) it can be used in nonholonomic motion planning as a tool for local planning tasks. The final trajectory results from connecting sub-trajectories derived from each of the local planning.