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Normal forms and approximated feedback linearization in discrete time

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Abstract

The paper discusses approximated feedback linearization of nonlinear discrete-time dynamics which are controllable in first approximation and introduces two types of normal forms. The study is set in the context of differential/difference representations of discrete-time dynamics proposed in [Monaco, Normand-Cyrot, in: Normand-Cyrot (Ed.), Perspectives in Control, a Tribute to Ioan Doré Landau, Springer, Londres, 1998, pp. 191–205].

Keywords: Normal forms; Approximated linearization; Nonlinear discrete-time systems

1. Introduction

The idea of simplifying the nonlinearities of a given discrete-time dynamics through coordinates change and feedback, launched in [14] in continuous-time control theory finds its roots in Cartan's method of equivalence or Poincaré's normal forms [24]. It has been more recently further developed and renewed making reference to controlled dynamics (see [13,7,11,23] and the references therein). On these bases, stabilizing strategies for dynamics with bifurcations have been proposed in [12]. While the approach can be similarly developed for both cases of vector fields (differential dynamical systems) and maps (discrete-time systems) [24], such a parallelism becomes difficult when dealing with forced dynamical systems. Even if many analogies can be set, differentiated studies are necessary. In discrete time most of the contributions are concerned with quadratic or cubic normal forms as this is in general enough to characterize control properties: quadratic approximated feedback linearization under dynamic feedback is studied in [1], stabilization of systems with uncontrollable modes or bifurcations in [8], observer design for systems with unobservable modes in [3]. In [15], quadratic and cubic normal forms are introduced to propose a systematic classification of discrete-time bifurcations taking place at equilibria due to loss of linear stabilizability. Following [15], homogeneous normal forms of degree m have been proposed in [9] for dynamics with controllable linear part.

With respect to these contributions [1,8,15,9], the problem is presently set and solved for dynamics controllable in first approximation in the formalism of differential/difference representations of discrete-time dynamics proposed in [19]. Such a set up, which does not imply any loss of generality in the present context, makes it possible to give for the first time a quite complete answer to the problem: two types of normal forms are proposed; the generic case of degree m is solved; the invariants are introduced and their role is clarified for achieving approximated feedback linearization. The advantage of the proposed approach is even more striking when considering sampled dynamics as illustrated by the examples worked out throughout the paper.

The study is addressed step-by-step, through homogeneous approximations of increasing degree of the Taylor-like expansions of the dynamics, coordinates changes and feedbacks. For each degree of approximation, say m, writing down the so-called *homological equations* which must be solved for achieving linearization, *normal forms of*

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degree m containing all the nonremovable nonlinear terms are characterized. As in continuous time [11,23], two kinds of normal forms are introduced depending whether one privileges cancellation of the nonlinear terms in the drift (dual normal form) or in the control vector fields (Kang's normal form). Provided the linear part and lower degree terms are fixed, homogeneous normal forms at a fixed degree are unique modulo homogeneous transformations of the same degree and a set of polynomials which are invariants under homogeneous transformations, the so called *homogeneous invariants*, are defined. The nullity of these invariants characterizes homogeneous feedback linearization at a fixed degree. It must be stressed that the normal forms here developed are different from those introduced in previous work for discrete-time dynamics in the form of maps. Preliminary results were given in [22] and in [21] with reference to quadratic approximations.

The paper is organized as follows. Section 2 is devoted to define the context and set the problem. Homogeneous transformation, feedback and feedback linear equivalences are formulated in the proposed differential/difference set up. Sections 3 and 4 contain the results. Homogeneous feedback and feedback linear equivalences at degree m are characterized either through the solvability of the homological equations of degree m or the nullity of the invariants. Merging the results, necessary and sufficient conditions ensuring approximated feedback equivalence are given. On these bases, two different types of homogeneous normal forms and extended normal forms are described in Section 4. Two examples are discussed. Notations are introduced in the sequel.

Notations: The state variables ζ and/or x belong to $\mathscr{X}$, an open set of R^n and the control variables v and/or u belong to $\mathscr{U}$, a neighborhood of zero in R. All the involved objects, maps, vector fields, control systems are analytic on their domains of definition, infinitely differentiable admitting convergent Taylor series expansions. A vector field on $\mathscr{X}$, analytically parameterized by u, $G(x,u) \in T_x\mathscr{X}$ defines a u-dependent differential equation of the form $\mathrm{d}x^+(u)/\mathrm{d}u = G(x^+(u),u)$ where the notation $x^+(u)$ indicates that the state evolution is a curve in R^n, parameterized by u. A R^n-valued mapping $F(.,u) : x \to F(x,u)$, denotes a forced discrete-time dynamics while $F : x \to F(x)$ and/or $F(.,0)$ denotes unforced evolutions. Given a generic map on $\mathscr{X}$, its evaluation at a point x is denoted indifferently by "(x)" or "$|_x$". $J_xF|_{x=0} = (\mathrm{d}F(x)/\mathrm{d}x)|_{x=0}$ indicates the Jacobian of the function evaluated at $x=0$. Given a vector field G on $\mathscr{X}$ and assuming that F is a diffeomorphism on $\mathscr{X}$, F_*G denotes the transport of G along F, defined as the vector field on $\mathscr{X}$ verifying $F_*G|_F = (J_xF)G$; analogously indicating by $F^p = F\circ\ldots\circ F$, the p-times composition of F, F_*^pG denotes the transport of G along F^p verifying $F_*^pG|_{F^p} = (J_xF^p)G$. The upperscript $(.)^{[m]}$ stands for the homogeneous term of degree m of the Taylor series expansion of the function or vector field into the parentheses. Analogously, $R^{[m]}(.)$ (resp. $R^{\geqslant m}(.)$) stands for the space of vector fields or functions whose components are polynomials (resp. formal power series) of degree m (resp. of degree $\geqslant m$) in the variables into the parentheses. The results are local in nature and convergence problems are not addressed so that the solutions proposed will be referred to as *formal* ones.

2. Context and problem statement

We consider throughout the paper a single-input discrete-time dynamics, $\zeta \to F(\zeta, v)$, which is controllable in first approximation around the equilibrium pair $(0,0)$. Without loss of generality as justified in the sequel, we make use of the differential/difference representation (DDR) introduced in [19] to describe such a dynamics; i.e. consider

$$\zeta^+ = F(\zeta), \tag{1}$$

$$\frac{\mathrm{d}\zeta^+(v)}{\mathrm{d}v} = G(\zeta^+(v), v); \quad \zeta^+(0) = \zeta^+, \tag{2}$$

where $G(.,v)$ admits the Taylor-type expansion around 0; $G(.,v) := G_1 + \sum_{i\geqslant 1}(v^i/i!)G_{i+1}$ with $G_1 := G(.,0)$; $G_{i+1} = (\partial^i G(.,v)/\partial v^i)|_{v=0}$ for $i \geqslant 1$; $F(0)=0$ and $G_1(0) \neq 0$.

To get more familiar with the representation (1–2), let the following comments.

- Provided completeness of the vector field $G(.,v)$, the associated flow is defined for any v, a nonlinear difference equation $\zeta \to F(\zeta, v)$ can be recovered integrating (2) between 0 and $v(k)$ with initialization at (1), $\zeta^+(0) = \zeta^+ = F(\zeta(k))$; we get

$$\zeta(k+1) = \zeta^+(v(k)) = F(\zeta(k), v(k)) = F(\zeta(k)) + \int_0^{v(k)} G(\zeta^+(w), w)\,\mathrm{d}w.$$

  An explicit exponential representation of $F(.,v)$ in terms of the G_i is given in [20].
- Reversing the arguments and starting from a difference equation $\zeta \to F(\zeta, v)$, the existence of (1–2) follows from the existence of $G(.,v)$ verifying $G(F(.,v),v)) = \partial F(.,v)/\partial v$. The invertibility of $F(.,0)$ is sufficient to prove that $G(.,v)$ can be locally uniquely defined as $G(.,v) := (\partial F(.,v)/\partial v)|_{F^{-1}(.,v)}$.
- The proposed formalism provides a new paradigm for modeling discrete-time as well as hybrid phenomena coupling continuous-time and discrete dynamics with jumps, switches and resets. It makes possible the complementary use of geometric and algebraic techniques so providing equivalent formalism and tools between continuous time and discrete time; a parallelism which is lost in the usual context of discrete-time dynamics in the form of maps as soon as nonlinear dynamics are concerned. Finally, let us note that the study of sampled dynamics can always be performed in such a context due to the invertibility of the drift under sampling.