Social Networks 44 (2016) 105–116

Contents lists available at ScienceDirect

Social Networks

journal homepage: www.elsevier.com/locate/socnet

# Latent space models for dynamic networks with weighted edges☆

Daniel K. Sewell[a], Yuguo Chen[b,*]

[a] Department of Biostatistics, University of Iowa, 145 N. Riverside Dr., Iowa City, IA 52242, USA
[b] Department of Statistics, University of Illinois at Urbana-Champaign, 725 S. Wright Street, Champaign, IL 61820, USA

ARTICLE INFO

*Keywords:*
Embedding
Markov chain Monte Carlo
Network dynamics
Visualization
Weighted network
Valued dyad

ABSTRACT

Longitudinal binary relational data can be better understood by implementing a latent space model for dynamic networks. This approach can be broadly extended to many types of weighted edges by using a link function to model the mean of the dyads, or by employing a similar strategy via data augmentation. To demonstrate this, we propose models for count dyads and for non-negative real dyads, analyzing simulated data and also both mobile phone data and world export/import data. The model parameters and latent actors' trajectories, estimated by Markov chain Monte Carlo algorithms, provide insight into the network dynamics.



## 1. Introduction

Representing relational data by networks is extremely useful and widely implemented. The dyadic relations which compose these networks are viewed as a set of actors and a set of edges between the actors. The edges can vary in many ways, such as being directed or undirected, static or temporal, binary or weighted. Binary networks, where between each actor an edge either does or does not exist, are encountered more often in the literature, although many such networks are by nature weighted. Weighted networks, also referred to as valued networks, consist of actors connected by edges which can take more than two values. By accounting for the weight, or strength, of the edges, the richness of the data can be better exploited. Examples of analyses of real world weighted networks include food webs (Krause et al., 2003), gene expression data (Zhang and Horvath, 2005), airline networks (Barrat et al., 2005), mobile phone networks (Onnela et al., 2007), and many more.

Often in binary networks it is of interest to compute various network measures, and recently there has been increasing work in extending these measures to weighted networks. Opsahl et al. (2010) derived for weighted networks measures for degree, closeness, and betweenness. Yang and Knoke (2001) derived a method for computing path length in the case of weighted edges. Opsahl and Panzarasa (2009) developed a method for analyzing the clustering that exists within a network with weighted edges. Other interesting works include Kunegis et al. (2009), which analyzed the case where edges took values in $\{-1, 0, 1\}$, and Newman (2004), which showed how to model networks whose edges are counts by representing them as multigraphs. To fully model the network, Krivitsky (2012) extended the commonly used exponential random graph model (ERGM) to account for networks whose dyads are counts; Krivitsky and Butts (2012) extended the ERGM to account for networks whose dyads are rankings.

Network data are most often inherently dynamic, even though it is frequently the case that the data are simply aggregated over time into one static network. Many popular static networks have been extended to longitudinal network data. Examples of this include the temporal exponential random graph model developed by Hanneke et al. (2010) and the separable temporal exponential graph model by Krivitsky and Handcock (2014), the mixed membership stochastic blockmodel for dynamic networks by Xing et al. (2010), and the latent space model for dynamic networks by several authors including Sarkar and Moore (2005), Sewell and Chen (2015b), Morgan (2014) and Durante and Dunson (2014).

This paper is focused on network data that is dynamic, weighted, and possibly directed. There are few resources available to the researcher investigating such data. Most approaches in existence focus on latent space models for dynamic undirected networks. Latent space models assume the dependence of the network is induced by a set of latent variables. Such approaches are typically intuitive and have the advantage of producing meaningful visualizations, allowing the researcher to better understand the network structure as well as the behavior of individual actors.

Sarkar et al. (2007) extended the CODE model of Globerson et al. (2004) for dynamic undirected networks. This method is

☆ This research was partly supported by the National Science Foundation grants DMS-1106796 and DMS-1406455.

* Corresponding author. Tel.: +1 217 244 3932; fax: +1 217 244 7190.
*E-mail address:* yuguo@illinois.edu (Y. Chen).

http://dx.doi.org/10.1016/j.socnet.2015.07.005
0378-8733/

an approximate filtering algorithm which models the longitudinal count networks, embedding the actors in a latent space. This method is not easily generalizable to other sorts of co-occurrence data besides counts, however, and cannot handle directed edges. Hoff (2011) described a multilinear model for undirected longitudinal networks. In this work, Hoff showed how to model undirected edges or ranked edges, where each dyad is an element from a finite ordered set, though it should be feasible to extend his approach to other types of dyads. Sewell and Chen (2015a) developed a latent space model for directed ranked dynamic networks, where each actor ranks each other actor, although it is not obvious how to extend this approach beyond this specific context.

The remainder of the paper is organized as follows. Section 2 extends the latent space model for dynamic networks with valued edges. Section 3 gives a method of estimation. Section 4 describes an approximation to reduce computational cost for large networks. Section 5 gives simulation results. Section 6 gives the results for analyzing a mobile phone network and world trade data. Section 7 provides a brief discussion.

## 2. Models

We assume here that each actor exists within some latent space which can be interpreted as a characteristic space, or a social space. When actors are closer together in this latent space, the probability of a stronger edge is increased (where a "stronger edge" means a stronger relationship, though the actual form of this is context specific).

We first introduce some general notation to be used throughout. Assume we have a set of actors $\mathcal{N}$ and a set of edges $\mathcal{E}$. Let $n = |\mathcal{N}|$ be the number of actors, and let $Y_t$ be the $n \times n$ adjacency matrix of the observed network at time $t$ whose entries $y_{ijt}$ correspond to the weight of the edge from actor $i$ to actor $j$ for $t \in \{1, 2, \ldots, T\}$. Let $\mathbf{X}_{it} \in \Re^p$ be the position vector of the $i$th actor at time $t$ within the $p$ dimensional latent space. Let $\mathcal{X}_t$ be the matrix whose $i$th row is $\mathbf{X}_{it}$. Finally, let $\boldsymbol{\Psi}$ be the vector of unknown parameters (which will vary depending on dyadic type).

As in Sarkar and Moore (2005) and Sewell and Chen (2015b), we assume the latent actor positions transition according to a Markov process, where the initial distribution is

$$\pi(\mathcal{X}_1|\boldsymbol{\Psi}) = \prod_{i=1}^{n} N(\mathbf{X}_{i1}|\mathbf{0}, \tau^2 I_p), \tag{1}$$

and the transition equation is

$$\pi(\mathcal{X}_t|\mathcal{X}_{t-1}, \boldsymbol{\Psi}) = \prod_{i=1}^{n} N(\mathbf{X}_{it}|\mathbf{X}_{i(t-1)}, \sigma^2 I_p), \tag{2}$$

for $t = 2, 3, \ldots, T$, where $I_p$ is the $p \times p$ identity matrix, and $N(\mathbf{x}|\boldsymbol{\mu}, \Sigma)$ denotes the multivariate normal probability density function with mean $\boldsymbol{\mu}$ and covariance matrix $\Sigma$ evaluated at $\mathbf{x}$. While this is the latent dependence structure used throughout the remainder of the paper, other dependence structures could be defined, such as the latent path model given by Morgan (2014).

In most dynamic network models it is assumed that the dependence structure of the network is fully induced by the latent positions of the actors. This assumption, along with the Markovian properties of the latent positions, leads to the state space temporal dependence structure given in Fig. 1, as well as the conditional independence of each dyad within a time period. The ranked networks of the form analyzed by Krivitsky and Butts (2012) and Sewell and Chen (2015a) are a counter example of where there is an extra dependency constraint in the data, but we will not discuss further these rare data types. What remains then is to derive an appropriate

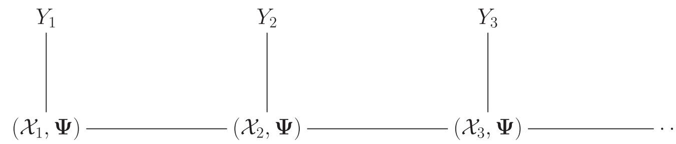


**Fig. 1.** Illustration of the dependence structure for the latent space model. $Y_t$ is the observed graph, $\mathcal{X}_t$ is the unobserved latent actor positions, and $\boldsymbol{\Psi}$ is the vector of model parameters.

conditional likelihood function, $\pi(Y_1, \ldots, Y_T|\mathcal{X}_1, \ldots, \mathcal{X}_T, \boldsymbol{\Psi}) = \prod_{t=1}^{T}\prod_{i \neq j}\pi(y_{ijt}|\mathcal{X}_t, \boldsymbol{\Psi})$.

Most latent space approaches have the conditional likelihood constructed by writing the logit of the edge probability as a linear form of covariates and a function of the latent variables, i.e., $\text{logit}(\pi(y_{ijt}|\cdot)) = \boldsymbol{\alpha}'\mathbf{w}_{ijt} + f_{\boldsymbol{\Psi}}(\mathbf{X}_{it}, \mathbf{X}_{jt})$, where $\boldsymbol{\alpha}$ is a vector of unknown parameters, $\mathbf{w}_{ijt}$ is a vector of dyad specific covariates, and $f_{\boldsymbol{\Psi}} : \Re^p \times \Re^p \to \Re$ is a function taking as its arguments two actors' latent variables. Our generalization of this has the basic form

$$g(\mathbb{E}(y_{ijt})) = \boldsymbol{\alpha}'\mathbf{w}_{ijt} + f_{\boldsymbol{\Psi}}(\mathbf{X}_{it}, \mathbf{X}_{jt}), \tag{3}$$

for some link function $g$. We can utilize the same types of link functions found in generalized linear mixed models. For example if our dyads are in the form of continuous data, we may set $g$ to be the identity; this may arise in, for instance, proximity networks (see, e.g., Olguın et al., 2009), where the distance between individuals is recorded on a regular basis. The common case of modeling binary dyads through the logit link function is yet another example. In Section 2.1 we will go into detail for the context of count data, using a log link function.

In some cases, however, the dyads cannot be modeled directly through a link function as in (3). Instead we can introduce additional latent variables, and then adopt a similar strategy. For example, we may consider a zero inflated model. The zero inflated model is a two component mixture model, where one could introduce additional latent indicator variables which determine whether the observation is coming from the component which is a point mass at zero or the component that has some other density function $\pi^*$ (e.g., $\pi^*$ is the Poisson density). We could then model $g(\mathbb{E}_{\pi^*}(y_{ijt}))$ as in (3). This situation may arise in large sparse weighted network data, such as company wide email count networks. Zero-inflated models are certainly not the only possibility of this type of data augmentation, as we will see in Section 2.2.

For the remainder of the paper we will focus on count data and non-negative continuous edges. We will furthermore utilize the conditional likelihood given by Sewell and Chen (2015b), determined by

$$f_{\boldsymbol{\Psi}}(\mathbf{X}_{it}, \mathbf{X}_{jt}) = \beta_{IN}\left(1 - \frac{d_{ijt}}{r_j}\right) + \beta_{OUT}\left(1 - \frac{d_{ijt}}{r_i}\right), \tag{4}$$

where $d_{ijt} = \| \mathbf{X}_{it} - \mathbf{X}_{jt} \|$ is the distance between actors $i$ and $j$ at time $t$ within the latent space, and $\boldsymbol{r} = (r_1, r_2, \ldots, r_n)$ is a vector of positive actor specific parameters constrained such that $\sum_{i=1}^{n} r_i = 1$ for model identifiability. For the remainder of the paper we will also, for simplicity, ignore the covariate information $\boldsymbol{\alpha}'\mathbf{w}_{ijt}$. It is straightforward to reincorporate such information into the work that follows.

Each $r_i$ can be thought of as the $i$th actor's social reach. That is, a larger value of $r_i$ implies that it is more likely for an edge, either $y_{i \cdot t}$ or $y_{\cdot it}$, to take a larger value. These $r_i$'s also hold a geometric interpretation within the latent space, specifically a radius. For example, in the context of binary networks, this radius can be understood to imply that actors inside of each other's radii have a greater than 1/2 probability of an edge, and actors outside of each