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# An exact convex relaxation of the freeway network control problem with controlled merging junctions

Marius Schmitt*, John Lygeros

*Automatic Control Laboratory, ETH Zurich, Physikstrasse 3, Zurich 8092, Switzerland*

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ABSTRACT

We consider the freeway network control problem where the aim is to optimize the operation of traffic networks modeled by the cell transmission model via ramp metering and partial mainline demand control. Optimal control problems using the cell transmission model are usually non-convex, due to the nonlinear fundamental diagram, but a convex relaxation in which demand and supply constraints are relaxed is often used. Previous works have established conditions under which solutions of the relaxation can be made feasible with respect to the original constraints. In this work, we generalize these conditions and show that the control of flows into merging junctions is sufficient to do so if the objective is to minimize the total time spent in traffic. We derive this result by introducing an alternative system representation. In the new representation, the system dynamics are concave and state-monotone. We show that exactness of the convex relaxation of finite horizon optimal control problems follows from these properties. Deriving the main result via a characterization of the system dynamics allows one to treat arbitrary monotone, concave fundamental diagrams and several types of control for merging junctions in a uniform manner. The derivation also suggests a straightforward method to verify if the results continue to hold for extensions or modifications of the models studied in this work.



## 1. Introduction

We study the freeway network control (FNC) problem, that is, the problem of optimal operation of freeway traffic for networks modeled using a variant of the cell transmission model (CTM) (Daganzo, 1994; 1995), with the standard objective of minimizing the total time spent (TTS). The CTM is a first-order traffic model obtained as a discretization of the kinematic wave model (Lighthill and Whitham, 1955; Richards, 1956). It describes road traffic by a linear conservation law and the nonlinear fundamental diagram, which models the relationship between traffic flow and traffic density. Finite-horizon optimal control problems for systems modeled by the CTM lead to nonconvex optimization problems in general, due to the nonlinear fundamental diagram. However, a convex optimization problem is obtained if the demand and supply constraints encoded in the fundamental diagram are relaxed. In particular, a linear program (LP) is obtained if a triangular or trapezoidal fundamental diagram is used (Ziliaskopoulos, 2000). In general, an optimal solution of the relaxed problem does not satisfy the dynamics of the CTM, but subsequent work has identified conditions for which the relaxation yields solutions to the original problem. In particular, it turns out that for a freeway segment with only onramp junctions and off-ramp

* Corresponding author.
*E-mail addresses:* schmittm@control.ee.ethz.ch (M. Schmitt), jlygeros@control.ee.ethz.ch (J. Lygeros).

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junctions solutions to the relaxed problem are feasible with respect to the CTM dynamics (Gomes and Horowitz, 2006). This result relies on the assumption that onramps are metered and inflow from onramps is not obstructed by mainline congestion, while off-ramps are assumed to be uncongested and hence, they do not obstruct mainline flow via congestion spillback. In Gomes et al. (2008) it has been shown that the corresponding CTM model is in fact a discrete-time, monotone system, a generalization of monotone maps (Hirsch and Smith, 2005) to systems with inputs. Basic definitions and results on monotone systems are presented in Angeli and Sontag (2003) for the analogous continuous-time case.

It is natural to ask whether monotonicity properties can be leveraged to facilitate the analysis and control of systems based on the CTM. However, it turns out that first-in, first-out (FIFO) diverging junctions as used in the CTM are not monotone (Coogan and Arcak, 2015). Alternative models for diverging junctions with monotone dynamics have been suggested (Lovisari et al., 2014a; 2014b), but these models do not preserve the turning rates. In Coogan and Arcak (2016) it is shown that the CTM dynamics satisfy a mixed-monotonicity property instead, but it is also suggested that the non-monotone dynamics of FIFO diverging junctions are exactly what dynamic traffic control should target in order to realize improvements over the uncontrolled case. Monotonicity has also been used to analyze robustness of optimal trajectories (Como et al., 2016). In addition, traffic routing problems have been considered (Como et al., 2013b; 2013a; 2015). In such problems, the turning rates are not fixed a priori, but they are (partially) actuated variables instead. With time-varying turning rates, diverging junctions do not exhibit FIFO dynamics, allowing one to circumvent the issues arising from the non-monotone effects. In particular, monotone routing policies are resilient to non-anticipated capacity reductions in individual links (Como et al., 2013b; 2013a). Subsequently, a class of distributed, monotone routing policies was proposed, that make use of the implicit back-propagation of congestion to stabilize maximal-throughput equilibria (Como et al., 2015).

It has also been suggested that solutions to the relaxed FNC problem (using relaxed demand and supply constraints) for arbitrary networks can be made feasible if traffic demand control is available in every cell of the CTM, for example via variable speed limits (Muralidharan and Horowitz, 2012; Como et al., 2016). However, it is questionable whether the assumption of demand control in every cell is realistic, in particular for freeway networks. Even if variable speed limits are implemented, possible operation modes are usually restricted, with only few distinct speed limits to choose from and constraints on how often these may change. Therefore, a crucial question is whether demand control in every cell is necessary to achieve the optimal cost of the relaxed problem, or if, for example, ramp metering is sufficient to do so. A partial answer is known for the special case of a symmetric triangular fundamental diagram in which the congestion wave speed is equal to the free-flow velocity in every cell. In this case, the solution to the relaxed FNC problem can be made feasible by using priority control for flows into merging junctions (Como et al., 2016, Proposition 2).

In this work, we generalize these results and consider CTM networks with FIFO diverging junctions and concave (but not necessarily symmetric or even piece-wise affine (PWA)) fundamental diagrams. We show that if the objective is to minimize the TTS, control of merging flows is sufficient to achieve the same cost as in the relaxed problem. This result allows us to use the convex, relaxed problem to efficiently compute solutions of the original nonconvex FNC problem. The main result of this work relies on the analysis of a novel, alternative system representation of the CTM. It turns out that the system dynamics are concave and state-monotone in the new representation. This allows us to employ results originally derived for convex, monotone systems (Rantzer and Bernhardsson, 2014) to show equivalence of the convex relaxation to the nonconvex optimal control problem. We generalize existing results, in particular (Como et al., 2016) where a related problem is addressed, in the following ways:

- Our main result applies to CTM networks with general concave, monotone fundamental diagrams. The existing result holds only for affine demand and supply functions with *identical* slopes (of opposite sign),[1] i.e., the case when the free-flow speed is equal to the congestion wave speed. Real-world free-flow speeds are typically significantly larger than congestion wave speeds.
- The main result is based on a novel system reformulation, in which the system dynamics are state-monotone and concave. The reformulation of the system dynamics links the result to properties of the dynamical system itself and suggests a straightforward method to verify if the results continue to hold for extensions or modifications of the models studied in this work.

This paper is structured as follows: in Section 2, we introduce results on the optimal control of concave, state-monotone systems that will be used subsequently. The freeway network model is introduced in Section 3. In Section 4, we perform a transformation of the system equations to derive an equivalent system representation and show that it is concave and state-monotone. This allows us to prove the main result of this work: the derivation of an exact, convex relaxation of the FNC problem for networks with controlled merging junctions. We contrast the dynamics of merging and diverging junctions in the original system model with the alternative representation in Section 5 in order to demonstrate the applicability and limitations of our results. In Section 6, we apply the main result to compute optimal open-loop control inputs for two freeway network examples, a real world freeway and an artificial freeway network designed to showcase the behavior of merging and diverging junctions. In Section 7, we summarize our contributions and provide suggestions for future work.

[1] A different result (Como et al., 2016, Proposition 1) allows for more general fundamental diagrams, but it requires demand control in *every* cell, as opposed to only in cells immediately upstream of merging junctions.