Acta Mechanica Solida Sinica, Vol. 20, No. 3, September, 2007 ISSN 0894-9166
Published by AMSS Press, Wuhan, China. DOI: 10.1007/s10338-007-0730-0

EXACT SOLUTION FOR A TWO-DIMENSIONAL LAMB'S PROBLEM DUE TO A STRIP IMPULSE LOADING *

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Received 22 December 2006, revision received 31 August 2007

ABSTRACT By applying the integral transform method and the inverse transformation technique based upon the two types of integration, the present paper has successfully obtained an exact algebraic solution for a two-dimensional Lamb's problem due to a strip impulse loading for the first time. With the algebraic result, the excitation and propagation processes of stress waves, including the longitudinal wave, the transverse wave, and Rayleigh-wave, are discussed in detail. A few new conclusions have been drawn from currently available integral results or computational results.

KEY WORDS Lamb's problem, exact solution, integral transform, strip impulse loading, excitation and propagation of elastic waves

I. INTRODUCTION

Since the initiating work of Lamb[1], many researchers have been known to devote themselves to exploring analytical solutions for the problems of instant wave propagation in elastic half-space subjected to impulsive loadings (these problems are called Lamb's problems currently). In the 1930s, Cagniard[2] developed Lamb's formulation into a general method for applying Fourier transform on the spatial variables and Laplace transform on the time variable. The operation of the two integrals in sequence then became the identity operation with no need of integration . Further modifications were provided by de Hoop[3] later so that it is usually referred to as Cagniard-de Hoop method. This method was adopted by Pekeris[4] and Chao[5] to obtain analytical solutions with vertical and plane loadings, respectively. Johnson[6] also applied Cagniard-de Hoop method to present a solution to the Lamb's problem of point-source, with a derivative form (derivative of Green function) about the origin and observed points. These analytical solutions were convenient for numerical analysis. Analysis on Lamb's problems with point source was further developed by Lapwood, Dix[7], Pinney[8], Pekeris and Lifson[9,10], Aggarwal and Ablow[11] etc. In 1983, Wang and Wang[12] deduced expressions for the surface displacement of the half-space subjected to a vertical point impact. Based on the fundamental solution in the complete space, Zheng[13] deduced the fundamental solutions of elastic half-space exerting the superposition principle, also for the point source case. However, algebraic-form exact solutions are seldom, if ever, found[2,3,9,12,14], because of mathematical difficulties. Moreover, infinite values were usually introduced into those solutions, because the problems were investigated with the load acting on a point or an infinite line or varying with time as a Dirac function.

* Project supported by the National Natural Science Foundation of China (No.10572002).

In view of the fact that a finite load acting on a finite pressure-area would represent the physical state more practically, the present paper investigates a two-dimensional Lamb's problem due to an impact loading acting on a finite strip area of the surface. The impact loading varies with time as a Heaviside function, with unloading counted in. Takemiya et al.[15] employed the Green function method to obtain an analytical result with integral expressions. And a problem still remains since numerical integration will bring error to the result in the end. In addition, computation is inadvisable for this problem because there are discontinuities of the derivatives which often introduce violent vibration or large errors into the computing process. The authors have obtained an exact algebraic solution for the surface stress of this problem for the first time using the integral transform method and the corresponding inverse transformation technique based upon the two types of integration. The piecewise expressions of the solution provide the information about all stress wave components to the corresponding branch points or singular points, so that the information is directly presented. Based on the resulting expressions, the excitation and propagation processes of stress waves, including the longitudinal wave, the transverse wave, and Rayleigh wave, are discussed in detail and shown by graphs, followed by some new conclusions never before drawn from those integral results or computational results of this problem.

II. BASIC FORMULATIONS

In the Cartesian coordinate system (x, y), considering a plane stress problem in two-dimensional half-space $-\infty < x < \infty$, $y \geq 0$. A vertical loading acts on the free edge $y = 0$, and is distributed equally from $(-a,\ 0)$ to $(a,\ 0)$. For the symmetry with respect to y-axis, the analysis can be restricted to $x \geq 0$, as well as $y \geq 0$. Then, the relevant wave field is governed by the following motion equations under the prescribed initial and boundary conditions:

$$\left(\nabla^2 - \frac{1}{c_l^2}\frac{\partial^2}{\partial t^2}\right)\varphi = 0, \quad \left(\nabla^2 - \frac{1}{c_t^2}\frac{\partial^2}{\partial t^2}\right)\psi = 0 \tag{1}$$

$$\varphi = \frac{\partial \varphi}{\partial t} = 0, \quad \psi = \frac{\partial \psi}{\partial t} = 0 \qquad (t = 0) \tag{2}$$

$$\lim_{y\to\infty}\left(\varphi,\ \psi,\ \frac{\partial \varphi}{\partial x},\ \text{etc.}\right) = 0 \qquad (0 \leq x,\ t > 0) \tag{3}$$

$$\sigma_y = -P_0 H(a - x) H(t_0 - t), \quad \tau_{xy} = 0 \qquad (y = 0) \tag{4}$$

where σ_x, σ_y and τ_{xy} are the stress components, φ and ψ are the Lamé potential functions, H is the Heaviside unit function, x and t are the spatial and time variables, respectively, t_0 the loading time, a the half length of the loading strip, and P_0 the magnitude of the loading. c_l and c_t are the speeds of longitudinal and transverse waves, μ and ν the elastic constant and the Poisson's ratio of the medium, respectively. According to Hooke's law, the stress components σ_x, σ_y and τ_{xy} are related to the strains, represented with the derivatives of the Lamé potentials φ and ψ, as follows

$$\begin{aligned}
\frac{\sigma_x}{2\mu} &= \frac{\partial^2 \varphi}{\partial x^2} + \frac{\nu}{1-\nu}\nabla^2\varphi + \frac{\partial^2 \psi}{\partial x \partial y} \\
\frac{\sigma_y}{2\mu} &= \frac{\partial^2 \varphi}{\partial y^2} + \frac{\nu}{1-\nu}\nabla^2\varphi - \frac{\partial^2 \psi}{\partial x \partial y} \\
\frac{\tau_{xy}}{2\mu} &= \frac{\partial^2 \varphi}{\partial x \partial y} + \frac{1}{2}\left(\frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2}\right)
\end{aligned} \tag{5}$$

III. ANALYTIC SOLUTION

Application of the Laplace transform (denoted by an over-bar) with parameter s over time t leads Eqs.(1) to the following two ordinary differential equations

$$\begin{aligned}
\nabla^2\overline{\varphi} &= \left(\frac{s}{c_l}\right)^2 \overline{\varphi} \\
\nabla^2\overline{\psi} &= \left(\frac{s}{c_t}\right)^2 \overline{\psi}
\end{aligned} \tag{6}$$