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Buckling Analysis of Axially Functionally Graded and Non-Uniform Beams Based on Timoshenko Theory**

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ABSTRACT In this paper, the buckling behaviors of axially functionally graded and non-uniform Timoshenko beams were investigated. Based on the auxiliary function and power series, the coupled governing equations were converted into a system of linear algebraic equations. With various end conditions, the characteristic polynomial equations in the buckling loads were obtained for axially inhomogeneous beams. The lower and higher-order eigenvalues were calculated simultaneously from the multi-roots due to the fact that the derived characteristic equation was a polynomial one. The computed results were in good agreement with those analytical and numerical ones in literature.

KEY WORDS buckling, axially functionally graded tapered beams, Timoshenko beam theory, coupled governing equations

I. Introduction

Non-homogeneous structural components with axially varying material properties are common in buildings and bridges as well as in machine parts. One of the examples is the functionally graded beam, the material properties of which continuously and gradually change in one or multi spatial directions via a technological manufacturing process aiming at optimizing these characteristics. The functionally graded materials have excellent performance in real world applications, e.g. the retained mechanical properties of each material constituent without the stress concentration involved because of the absence of distinct interfaces, making such new materials very promising in thermal and structural fields. Another application is in the beams or rods with non-uniform cross sections, where the heterogeneity is driven by varying dimensions.

A substantial body of research has been carried out on the buckling behaviors of such elastic structures. In order to simplify the problem, the Euler-Bernoulli beam model is consistently used for discussing the buckling of non-uniform beams. In general, it is difficult to seek analytic solutions to the critical buckling loads of non-uniform columns because the governing equation has variable coefficients. Fortunately, with the transverse displacement and the stiffness of the axial direction chosen as some polynomials, Elishakoff and co-workers used the inverse method to find the exact solutions to the critical buckling loads for nonhomogeneous beams[1–4]. However, the inverse method cannot be applied to graded beams of any axial inhomogeneity and boundary conditions. As a result, many approximate methods or numerical

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techniques have been formulated to deal with the buckling of axially inhomogeneous beams via the Euler-Bernoulli theory, including the Rayleigh's quotient method[5], the finite element method[6–8], the finite difference method[9], the boundary element method[10], the integral equation method[11], the differential quadrature method[12], and others[13–15]. Based on the Euler-Bernoulli beam theory, Shahba and Rajasekaran have investigated the free vibration and stability of tapered and axially functionally graded beams using the differential transform element method[16].

It is well-known that the Euler-Bernoulli theory of beams neglects the effects of rotary inertia and shear deformation of cross sections. As a result, the critical buckling load is always overestimated based on the Euler-Bernoulli theory, which is only applicable to slender beams. By taking the shear deformation and rotary inertia into account, the Timoshenko beam theory gives a more accurate model. However, the application of Timoshenko theory in describing the buckling behaviors of axially inhomogeneous beams may cause difficulty in mathematical treatment due to the fact that the governing equations are the two coupled differential equations with variable coefficients. Some analytical and numerical approaches have been suggested to dealing with the dynamic behaviors of axially non-homogeneous Timoshenko beams. In Refs.[17–19], the step-reduction method, the Rayleigh-Ritz method and the differential transform method have been respectively used to investigate the free vibration of axially non-uniform beams based on Timoshenko theory. For buckling of axially non-homogeneous Timoshenko beams, Shahba et al. used the finite element approach to analyze the effects of taper ratio, elastic constraint, attached mass and material non-homogeneity on the critical buckling load[20]. Recently, by means of the dynamic stiffness method, Rajasekaran[21] has investigated the buckling and vibration of axially functionally graded non-uniform Timoshenko beams through differential transformation.

The objective of this study is to present a new and simple approach to the buckling problem of Timoshenko beams with the material properties varying arbitrarily in the axial direction. To avoid solving the coupled differential equations directly, the governing coupled equations with variable coefficients are converted into the characteristic polynomial equations for different types of end supports by means of the auxiliary function and power series. In the numerical discussion, the introduced method is used to evaluate the buckling loads of homogeneous beams and axially graded non-uniform beams with polynomial and exponential properties.

II. Basic Equations

An axially non-homogeneous Timoshenko beam of length L is considered, the material and cross-sectional properties of which are assumed to vary arbitrarily in the length direction. When Timoshenko theory is adopted, the two coupled governing differential equations of transverse deflection w and rotation θ for buckling of an inhomogeneous beam can be written as[20,21]

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(E(x)I(x)\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)+\kappa G(x)A(x)\left(\frac{\mathrm{d}w}{\mathrm{d}x}-\theta\right)=0 \quad (0\le x\le L) \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\kappa G(x)A(x)\left(\frac{\mathrm{d}w}{\mathrm{d}x}-\theta\right)\right]-P\frac{\mathrm{d}^2w}{\mathrm{d}x^2}=0 \quad (0\le x\le L) \tag{2}$$

where x, κ and P are the axial coordinate, the shear correction factor and the axial compressive load along the centroidal axis, respectively. In this paper, the bending stiffness $E(x)I(x)$ and shear stiffness $G(x)A(x)$ are both dependent on the axial coordinate x. It should be noted that $\widetilde{E}(x)=E(x)I(x)$ and $\widetilde{G}(x)=\kappa G(x)A(x)$ are given functions, while $w(x)$ and $\theta(x)$ are unknown functions. The key to this problem is to determine the unknown eigenvalue P, which is referred to as the critical buckling loads.

Since the critical buckling loads are closely related to the end supports of the beam, it is instructive to give expressions for the relevant familiar end conditions such as the simply supported, clamped, free