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LONGITUDINAL VIBRATIONS OF ARBITRARY NON-UNIFORM RODS**

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ABSTRACT Free and steady-state forced longitudinal vibrations of non-uniform rods are investigated by an iteration method, which results in a series solution. The series obtained are convergent and linearly independent. Its convergence is verified by convergence tests, its linear independence confirmed by the nonzero value of the corresponding Wronski determinant. Then, the solution obtained is an exact one reducible to a classical solution for the case of uniform rods. In order to verify the method, two examples are presented as an application of the proposed method. The results obtained are equivalent to the method in literature. In contrast to the proposed method capable of dealing with arbitrary non-uniform rods in principle, the method in literature is confined to work on special cases.

KEY WORDS longitudinal vibration, arbitrary non-uniform rod, series solution, convergence

I. INTRODUCTION

The vibration of non-uniform rods is a subject of considerable scientific and practical interest that has been studied extensively. A study of vibration of tapered rods indicates that the natural frequencies are only affected slightly by the taper[1]. It was shown that the governing equation of tapered rods could be reduced to the form of a wave equation by a change of variables[2]. In seeking exact solutions for the problem, Abrate[3] obtained a closed form solution for rods whose cross-sectional area varies as $A\left(x\right)=A_0\left(1+ax/L\right)^2$. Exact solutions also exist for exponential and catenoidal rods[4]. Using appropriate transformations Kumar and Sujith[5] solved the problem with cross-sectional area $A\left(x\right)=A_0\left(a+bx\right)^n$ and $A\left(x\right)=A_0\sin^2(a+bx)$. In a recent study Anil and Sujith[6] solved the problem with cross-sectional area $A(x)=kx^n\mathrm{e}^{bx^2}$ and $A(x)=kx^n\mathrm{e}^{bx}$. Guo and Yang[7] also investigated the problem with cross-sectional area $A(x)=A_0\mathrm{e}^{ax+bx^2}$ using Kummar functions and the WKB (Wentzel-Kramers-Brillouin) method. Recently, the wave motion in non-uniform one-dimensional waveguides[8–10] has been studied. However, to the best of the authors' knowledge, there is no exact solution of longitudinal vibration for arbitrary non-uniform rods in literature.

In this paper, an iteration method is proposed to solve the free and steady-state longitudinal vibration of arbitrary non-uniform rods, which results in a series solution. Its convergence is verified by convergence

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tests. In §II, the governing equations and the proposed method are introduced. §III investigates the uniform rods using the proposed method. §IV deals with the case of arbitrary non-uniform rods. The frequency equation and mode function of free vibration are obtained in §V, and two examples are presented. Conclusions are drawn in §VI.

II. THE GOVERNING EQUATION AND THE PROPOSED METHOD

The relevant non-uniform rod is shown in Fig.1. Its general governing equation of longitudinal vibrations reads

$$\frac{\partial}{\partial x}\left[E_0 A_0 f_1(x)\frac{\partial u}{\partial x}\right] = \rho_0 A_0 f_2(x)\frac{\partial^2 u}{\partial t^2} - p(x,t) \tag{1}$$

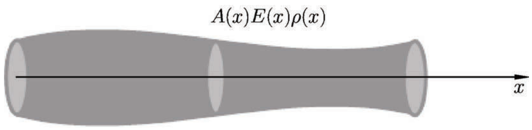

Fig. 1. A non-uniform rod.

where A_0, ρ_0 and E_0 are the nominal area, the nominal density and the nominal Young's modulus, respectively; u is the displacement; the exciting force $p(x,t)$ imposed on the rod is assumed to be harmonic $p(x,t) = p(x)\,\mathrm{e}^{\pm \mathrm{i}\omega t}$. The Young's modulus, the density and the cross-sectional area vary with $E(x)\,A(x) = E_0 A_0 f_1(x)$ and $\rho(x)\,A(x) = \rho_0 A_0 f_2(x)$. If the Young's modulus $E(x)$ or the density $\rho(x)$ varies with the longitudinal coordinate x and the cross-sectional area constant, it's the so called axial functionally graded material. If the Young's modulus and density remain constant, while the cross-sectional area varies along the longitudinal direction, it's the non-uniform rod with variable cross section. The two cases can both be described by the two functions $f_1(x)$ and $f_2(x)$. By neglecting the time factor $\mathrm{e}^{\pm \mathrm{i}\omega t}$, the governing equation (1) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f_1(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right] + \beta^2 f_2(x)\,u(x) = -\frac{p(x)}{E_0 A_0} \tag{2}$$

where $\beta = \omega\sqrt{\rho_0/E_0}$.

In this paper, the functions $f_1(x)$, $f_2(x)$ and $p(x)$ satisfy the following assumptions:
(a) the function $f_1(x)$ is positive, continuously differentiable and bounded;
(b) the function $f_2(x)$ is positive, bounded and piecewise continuous;
(c) the function $p(x)$ is bounded and piecewise continuous.

Although the assumptions are strong and can be weakened in mathematics, they are consistent with the common case in mechanics. It can be found from the assumptions that

$$0 < f_{1\min} \le f_1(x) \le f_{1\max}, \quad 0 < f_{2\min} \le f_2(x) \le f_{2\max}, \quad |p(x)| \le p_{\max} \tag{3}$$

It's well known that constructing the general solution of Eq.(2) in mathematics is no easy job. Here the solution form of Eq.(2) is assumed to be

$$u(x) = \sum_{n=0}^{\infty} u_n(x)\,\beta^{2n} = u_0(x) + u_1(x)\,\beta^2 + u_2(x)\,\beta^4 + \cdots \tag{4}$$

Substituting Eq.(4) into Eq.(2) and letting the coefficients of β vanish yields the recursive equation as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f_1(x)\frac{\mathrm{d}u_0(x)}{\mathrm{d}x}\right] = -\frac{p(x)}{E_0 A_0}, \quad \frac{\mathrm{d}}{\mathrm{d}x}\left[f_1(x)\frac{\mathrm{d}u_n(x)}{\mathrm{d}x}\right] + f_2(x)\,u_{n-1}(x) = 0 \quad (n = 1,2,3\cdots) \tag{5}$$

The general solution of $u_0(x)$ and $u_n(x)$ in Eq.(5) reads

$$u_0(x) = \int_0^x \left[\frac{1}{f_1(t_1)}\int_0^{t_1} -\frac{p(t)}{E_0 A_0}\mathrm{d}t\right]\mathrm{d}t_1 + c_1\left[\int_0^x \frac{1}{f_1(t)}\mathrm{d}t\right] + c_0 \tag{6}$$