Computers & Industrial Engineering 117 (2018) 6–18

Contents lists available at ScienceDirect

Computers & Industrial Engineering

journal homepage: www.elsevier.com/locate/caie

Dynamic Economic Lot-Sizing Problem: A new $O(T)$ Algorithm for the Wagner-Whitin Model

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ARTICLE INFO

Keywords:
Inventory
WW algorithm
Linear-time complexity
Economic lot-sizing
Single machine batch-sizing problem

ABSTRACT

Wagner and Whitin (1958) develop an algorithm to solve the dynamic Economic Lot-Sizing Problem (ELSP), which is widely applied in inventory control, production planning, and capacity planning. The original algorithm runs in $O(T^2)$ time, where T is the number of periods of the problem instance. Subsequently, other researchers develop linear-time algorithms to solve the Wagner-Whitin (WW) lot-sizing problem; examples include the ELSP and equivalent Single Machine Batch-Sizing Problem (SMBSP). This paper revisits the algorithms for the ELSP and SMBSP under WW cost structure, presents a new efficient linear-time algorithm, and compares the developed algorithm with equivalent algorithms in the literature. The developed algorithm employs a lists and stacks data structure, which is a completely different approach than that of the comparable algorithms for the ELSP and SMBSP. Analysis of the developed algorithm shows that it executes fewer different actions throughout and hence it improves execution time by a maximum of 51.40% for the ELSP and 29.03% for the SMBSP.

1. Introduction

The Economic Lot-Sizing Problem (ELSP) is an important issue in production and inventory control. Typically, a product is created or purchased in batch quantities and placed in stock. As the stock is depleted, more production or procurement must take place to replenish it. The main objective of the ELSP is to determine an optimum production or replenishment policy for a manufacturing or inventory system to meet the required market demand with the least possible expenditure. This policy decision is crucial, so it is a matter of interest for many researchers. Harris (1913) introduces his well-known and fundamental Economic Order Quantity model, in which he assumes demand to be a continuous function over time. However, Wagner and Whitin (1958) provide a different approach to solving the lot-sizing problem. They consider time in discrete periods and assume that demand in each period is known in advance.

Wagner and Whitin (1958) develop a forward recursion algorithm to obtain a minimum total cost inventory management scheme, which satisfies demand known a priori in every period. They consider uncapacitated (i.e., without bounds on production and inventory) lot-sizing problems for a single-item inventory system. Their algorithm's main assumption is that an item produced in a period can satisfy the demand in that and subsequent periods. Any item incurs setup and unit production costs, and any item carried to the next period incurs a unit inventory holding cost. The goal is to find a minimum cost production plan. The Wagner-Whitin (WW) algorithm runs in $O(T^2)$ time, where T is the number of periods of the problem instance. Wagelmans, Hoesel, and Antoon (1992) develop a linear-time algorithm (based on a geometric approach) for special cases of the WW problem where production and holding costs remain constant. Aggarwal and Park (1993) identify that the ELSP gives rise to Monge arrays (a special type of 2×2 array in which the four elements at the intersection points are such that the sum of the upper-left and lower-right elements across the main diagonal is less than or equal to the sum of the lower-left and upper-right elements across the antidiagonal). Employing the properties of a Monge array, Aggarwal and Park provide a linear-time algorithm for the WW problem. Albers and Brucker (1993) study the complexity of the single machine batch-sizing problem (SMBSP) and develop an algorithm for the shortest path problem that can be solved in linear time. The SMBSP can be defined as follows. Suppose there are n jobs, with given processing times, to be processed in batches on one machine. A batch is a set of jobs that is processed together. The number of jobs in a batch is called the *batch size*. The production of a batch requires machine setups, which are assumed to be both sequence- and machine-independent. The problem is to find the optimal batch size that minimizes the total flow time. Flow time of a batch is the sum of the processing times of all jobs in that batch plus the machine setup time. Therefore, all jobs in a batch have the same flow time.

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https://doi.org/10.1016/j.cie.2018.01.010
Received 17 April 2017; Received in revised form 11 January 2018; Accepted 14 January 2018
0360-8352/

The Wagelmans et al. (1992) and Aggarwal and Park (1993) algorithms are famous in the field of ELSP and obtain excellent results in terms of time complexity. This paper revisits these algorithms and presents a new linear-time algorithm for the ELSP under WW cost structure. The developed algorithm employs a lists and stacks data structure, which is a completely different approach than that of the existing algorithms (Aggarwal & Park, 1993; Wagelmans et al., 1992) in the literature. We match our result with the other algorithms (Aggarwal & Park, 1993; Wagelmans et al., 1992) for the ELSP and find that the new algorithm takes less CPU time and performs fewer various operations. The ELSP is equivalent to the SMBSP (see Section 4), so the developed algorithm is also applicable for solving the SMBSP. The developed algorithm is compared with the Albers and Brucker (1993) algorithm for the SMBSP and demonstrates its superiority in terms of various metrics of comparison. For the ELSP, we assume that holding costs are stationary but setup costs are time variant. However, for the SMBSP, we assume that setup costs for every job are constant.

The rest of this paper is organized as follows. Section two reviews the related work in the literature. Section three provides a simpler linear-time algorithm for the WW dynamic program and its proofs. Section four illustrates how the developed algorithm can be implemented for the SMBSP. The fifth section presents a numerical example showing the implementation of the developed algorithm. Section six illustrates the computational results assessing the new algorithm's performance. Finally, the seventh section represents the conclusion.

2. Literature review

During the 1980s and 1990s, many researchers improve the computational complexity of the algorithms for the simple uncapacitated ELSP. Evans (1985) presents an efficient computer implementation of the WW algorithm, which is an $O(T^2)$ time dynamic programming recursion, where T denotes the number of periods. He exploits the special structure of the problem, which requires low core storage, enabling it to be potentially useful and efficient for solving lot-sizing problems.

There are many studies in the literature that discuss the improvement opportunities of the Wagner-Whitin algorithm to solve the single-item uncapacitated dynamic ELSP. Federgruen and Tzur (1991) develop a simple forward algorithm, which can be implemented in $O(TlogT)$ time and $O(T)$ space for the dynamic ELSP. They also provide linear-time algorithms for two distinct cases: (i) models without speculative motives for carrying stock and (ii) models with nondecreasing setup costs. Wagelmans et al. (1992) develop a backward dynamic programming recursion for the uncapacitated ELSP that runs in $O(T)$ time for the WW case and $O(TlogT)$ time for a more general case, where marginal production costs differ between periods and all cost coefficients are unrestricted in sign. Aggarwal and Park (1993) show that the dynamic programming formulation of the uncapacitated ELSP gives rise to the Monge array, and they prove that the structure of the Monge arrays can be exploited to obtain a significantly faster algorithm. They present an $O(TlogT)$ time algorithm for both basic and backlogging ELSPs when the production, inventory, and backlogging costs are linear, and they show that for the special case of the WW model, this algorithm runs in $O(T)$ time.

van Hoesel, Wagelmans, and Moerman (1994) also consider the Wagner and Whitin (1958) dynamic ELSP and generalize the algorithms developed by Federgruen and Tzur (1991) and Wagelmans et al. (1992) by introducing two basic geometric techniques to solve the ELSP in $O(TlogT)$ time. They prove that if there is no speculative motive to hold inventory, the ELSP can be solved in $O(T)$ time. They discuss the forward and backward recursions for lot-sizing problems and the extension to the model, which allows backlogging, lot-sizing with start-up costs, and a generalized version of the model with learning effects in setup costs. They also show that the techniques used by Federgruen and Tzur (1991) and Wagelmans et al. (1992) are essentially the same.

Albers and Brucker (1993) study the complexity of the SMBSP for a fixed job sequence and develop a backward recursion algorithm that runs in $O(n)$ time, where n denotes the number of jobs. Baki and Vickson (2003) consider a lot-sizing problem in which a single operator completes a set of n jobs requiring operations on two machines. They develop an efficient algorithm for minimizing maximum lateness that can be solved in $O(n)$ time for both open and flow-shop cases. Mosheiov and Oron (2008) address the SMBSP to minimize total flow time for bounded batch sizes. They assume identical processing time for all jobs and identical setup time for all batches and introduce an efficient solution approach for both cases of an upper and a lower bound on the batch sizes. Li, Ishii, and Masuda (2012) extend Mosheiov and Oron (2008) by introducing a flexible upper bound for batch sizes, with the objectives of maximizing customer satisfaction and minimizing maximum completion time and flow time.

Teksan and Geunes (2015) provide a polynomial-time algorithm for the dynamic ELSP with convex costs in the production and inventory quantities. They consider a classic discrete-time, finite-horizon, uncapacitated, single-stage, dynamic lot-sizing problem with no backlogging. The resulting time complexity of their algorithm is $O(T^2logT)$.

Archetti, Bertazzi, and Grazia Speranza (2014) investigate an uncapacitated ELSP with two different cost discount functions. The first is the modified all unit discount cost function, which is piecewise and linear. They show that the problem can be solved in $O(I^2T^3)$ time complexity, where I is the number of echelons and T is the length of the discrete finite horizon. The second is the incremental discount cost function, which is increasing, piecewise, and linear. They show that the ELSP can be solved using a more efficient polynomial algorithm with an $O(T^2)$ time complexity.

Akbalik and Rapine (2013) study the complexity of a single-item uncapacitated lot-sizing problem with batch delivery, focusing on the general case of time-dependent batch sizes. They allow incomplete batches (fractional batches) in their model, with known demand over the planning horizon. They do not allow backlogging. They establish that if the cost parameters (setup cost, fixed cost per batch, unit procurement cost, and unit holding cost) are allowed to be time dependent, the problem is NP hard. By contrast, if all cost parameters are stationary and no unit holding cost is assumed, the problem is polynomially solvable in $O(T^3)$ time. They also show that in the case of divisible batch sizes, the problem of time-varying setup costs can be solved in time $O(T^3logT)$ if there are no unit procurement or holding cost elements.

Wang, He, Sun, Xie, and Shi (2011) also study a single-item uncapacitated lot-sizing problem. They develop an $O(T^2)$ time algorithm to determine the lot sizes for manufacturing, remanufacturing, and outsourcing that minimizes the total cost, which consists of the holding costs for returns, manufactured and remanufactured products, setup, and outsourcing costs. Chu, Chu, Zhong, and Yang (2013) consider an uncapacitated single-item lot-sizing problem with outsourcing/subcontracting, backlogging, and limited inventory capacity. The backlogging level at each period is supposed to be limited. The authors show that this problem can be solved in $O(T^4logT)$ time. Fazle Baki, Chaouch, and Abdul-Kader (2014) discuss the ELSP with product return and remanufacturing and show that this kind of problem is NP hard. Retel Helmrich, Jans, van den Heuvel, and Wagelmans (2015) study the ELSP with an emission constraint. They show that ELSP with emission constraint is NP hard and propose several solution methods.

Hsu (2000) introduces an $O(T^4)$ time algorithm for the dynamic uncapacitated ELSP with perishable inventory under age-dependent holding costs and deterioration rates, where all cost functions are nondecreasing concave. Hsu (2003) extends Hsu (2000) by allowing backlogging in the model and gives an algorithm that runs in $O(T^4)$ time under some assumptions on cost functions and demand. Sargut and Işık (2017) extend Hsu (2003) by incorporating production capacity in the dynamic ELSP and provide a dynamic-programming-based heuristic for the solution of the overall problem.

Studies are ongoing to incorporate capacity constraints as an extension to the WW algorithm. Bitran and Yanasse (1982) show that Capacitated Lot-Sizing Problems (CLSPs) belong to the class of NP-hard