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A study of process monitoring based on inverse Gaussian distribution

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ABSTRACT

The inverse Gaussian distribution has considerable applications in describing product life, employee service times, and so on. In this paper, the average run length (ARL) unbiased control charts, which monitor the shape and location parameters of the inverse Gaussian distribution respectively, are proposed when the in-control parameters are known. The effects of parameter estimation on the performance of the proposed control charts are also studied. An ARL-unbiased control chart for the shape parameter with the desired ARL_0, which takes the variability of the parameter estimate into account, is further developed. The performance of the proposed control charts is investigated in terms of the ARL and standard deviation of the run length. Finally, an example is used to illustrate the proposed control charts.

1. Introduction

The control charts are used to monitor and detect the shifts in the process mean and variance of a quality characteristic of interest in a process. The usual Shewhart $\overline{X}$ and R control charts are based on the assumption that the distribution of the observed data is normal. However, there are many cases in which the normality assumption is not valid (Chan & Cui, 2003; Haridy & Ei-Shabrawy, 1996; Roes & Does, 1995). When the underlying distribution is skewed, there are potential problems, namely, the false alarm rates and detection power of an out-of-control condition often substantially differ from what we expect under the normal case (Mahoney, 1998; Tadikamalla, Banciu, & Popescu, 2008). One method used to compute control limits for skewed distributions is to first transform the data to make it quasi-normal and then use the traditional Shewhart control charts (Kao, 2010; Liu, Xie, Goh, & Chan, 2007; Tadikamalla & Popescu, 2007; Xie, Goh, & Tang, 2000). But it is not frequently used because it is difficult to explain the alarm intuitively. Moreover, when the form of the process distribution is known, one may use the exact method, not an approximate method, because the accurate control limits are more likely to detect whether a process is in control (Chan & Cui, 2003; Tadikamalla & Popescu, 2007). Meanwhile, numerous papers show that the performances of the control charts with asymmetric control limits are better than those with symmetric control limits when the underlying distribution is heavily skewed. For some other related discussion, see, e.g., Tagaras (1989); Quesenberry (1991); Quesenberry, 1995, Woodall (1997); Yazici and Kan (2009) and Chen and Kuo (2010).

The inverse Gaussian distribution is one of the most important skewed distributions because it is highly flexible and there are a few major advantages relative to the other positive skewed distributions (Chhikara & Folks, 1989; Hawkins & Olwell, 1997; Tian & Wilding, 2005; Wu & Li, 2012). Hawkins and Olwell (1997) further gave some motivations to design the control charts to monitor the changes in the parameters of the inverse Gaussian distribution. In the previous studies of the control charts for the inverse Gaussian distribution, Edgeman (1989a, 1989b) constructed the Shewhart-type control charts with the probability limits for detecting the changes in the mean and variability of the inverse Gaussian distribution. Aminzadeh (1993) proposed the control charts with the probability limits for monitoring the mean and dispersion of the inverse Gaussian distribution based on the approximate distribution of the monitoring statistics using the exponential smoothing technique. Based on the sequential probability ratio tests and cumulative sum plans, Edgeman (1996) considered the control chart to monitor the shifts in the location parameter assuming that the shape parameter of the inverse Gaussian distribution is known. Shankar and Joseph (1996) proposed a cumulative sum chart to monitor the mean of the inverse Gaussian distribution assuming that the shape parameter is known. Hawkins and Olwell (1997) developed a CUSUM control chart for the location parameter with the assumption of the fixed shape parameter, and proposed another CUSUM control chart for the shape parameter assuming that the location parameter is fixed. Sim (2003) developed the control chart with the probability limits for the variability under the assumption that the shape parameter

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is known, namely, the control chart is for the location parameter. Lio and Park (2010) proposed the parametric bootstrap control charts for monitoring the percentiles of the inverse Gaussian distribution.

For the skewed distributions, the mean and variance may be ineffective summary statistics of the process measure (Hawkins & Olwell, 1997), thus we propose the control chart for the shape parameter and the control chart for the location parameter when the shape parameter is in control. Moreover, the Shewhart-type control charts mentioned above are average run length (ARL) biased (For a control chart, if the ARLs under the out-of-control processes are uniformly smaller than that under the in-control process, it is called an ARL-unbiased control chart), which is common for data that follow the skewed distributions (Cheng & Chen, 2011; Guo & Wang, 2014; Zhang, Xie, & Goh, 2006). The ARL-biased problem is highly undesirable in practice, since it takes a longer time on average to signal the assignable causes than that when there is no assignable cause. This paper thus focuses on the design of the ARL-unbiased control charts.

This paper is organized as follows. In Section 2, we propose two statistics to monitor the shape and location parameters, respectively. In Section 3, we develop the procedures to design the ARL-unbiased control charts for the shape and location parameters, respectively. In Section 4, we study the effects of parameter estimation on those control charts given in Section 3. In Section 5, we propose a procedure to design the ARL-unbiased control chart with the desired ARL_0 for the shape parameter when the in-control shape parameter is estimated, and study the performance of the proposed control chart. Finally, we use an example to illustrate the proposed control charts.

2. Monitoring statistics

The probability density function (p.d.f.) of the inverse Gaussian distribution $IG(\mu,\lambda)$ is given by

$$f(x,\mu,\lambda)=\left(\frac{\lambda}{2\pi x^3}\right)^{1/2}\exp\left[\frac{-\lambda(x-\mu)^2}{2\mu^2 x}\right],\quad x>0, \tag{1}$$

where $\lambda>0$ is the shape parameter and $\mu>0$ is the location parameter. Its mean and variance are μ and μ^3/λ, respectively, therefore the coefficient of variation is $CV=(\mu/\lambda)^{1/2}$. The density is unimodal with shape depending only on $\theta=\lambda/\mu=1/CV^2$ (Edgeman, 1996).

Suppose that $X_1,X_2,\ldots,X_n$ is a random sample with size n from the $IG(\mu,\lambda)$ distribution.

Let

$$T_1(\lambda)=\lambda\sum_{i=1}^{n}\left[\frac{1}{X_i}-\frac{1}{\overline{X}}\right]\qquad\text{and}\qquad T_2(\mu,\lambda)=\frac{\overline{X}}{\mu},$$

where $\overline{X}=\frac{1}{n}\sum_{i=1}^{n}X_i$. Then $\overline{X}$ and $\sum_{i=1}^{n}X_i^{-1}$ are sufficient statistics for the parameters (λ,μ), and $T_1(\lambda), T_2(\mu,\lambda)$ are independent, and $T_1(\lambda)\sim\chi^2(n-1)$, $T_2(\mu,\lambda)\sim IG(1,n\theta)$ (Tweedie, 1957).

2.1. A monitoring statistic for λ

Let λ_0 be the in-control value of the shape parameter λ, and $\lambda_1=\rho\lambda_0$ is the true value of λ. If $\rho=1$, the parameter λ is in control, otherwise the parameter λ has shifted, namely, the parameter λ has decreased when $0<\rho<1$ or increased when $\rho>1$. Notice that the distribution of the statistic $T_1(\lambda)$ does not depend on the parameters λ and μ,

$$T_1(\lambda_0)=\lambda_0\sum_{i=1}^{n}\left[\frac{1}{X_i}-\frac{1}{\overline{X}}\right]$$

is thus proposed as a monitoring statistic of the parameter λ.

Since $T_1(\lambda_0)=T_1(\lambda_1)/\rho$, the value of the monitoring statistic $T_1(\lambda_0)$ becomes smaller on average when λ shifts from λ_0 up to $\lambda_1(>\lambda_0)$, as well as the value of $T_1(\lambda_0)$ becomes larger on average when λ shifts from λ_0 down to $\lambda_1(<\lambda_0)$. Thus it is reasonable to monitor the changes in λ using $T_1(\lambda_0)$.

2.2. A monitoring statistic for μ

Since the maximum likelihood estimation of the location parameter μ is $\overline{X}$ and the sample mean $\overline{X}\sim IG(\mu,n\lambda)$ or $\overline{X}/\mu\sim IG(1,n\lambda/\mu)$ when the sample comes from $IG(\mu,\lambda)$, similar to the strategy of studying the R or S chart first before considering the $\overline{X}$ chart in the normal case, the location parameter μ can be monitored only when the shape parameter λ is in control. In fact, in some physical applications it is natural to hold λ constant (Tweedie, 1957).

Let μ_0 be the in-control value of the location parameter μ, and $\mu_1=\delta\mu_0$ is the true value of μ. If $\delta=1$, the parameter μ is in control, otherwise the parameter μ has shifted, namely, the parameter μ has decreased when $0<\delta<1$ or increased when $\delta>1$. Since $T_2(\mu_0,\lambda_0)=\delta T_2(\mu_1,\lambda_0)$, the value of $T_2(\mu_0,\lambda_0)$ becomes larger on average when the location parameter μ shifts from μ_0 up to $\mu_1(>\mu_0)$, as well as the value of $T_2(\mu_0,\lambda_0)$ becomes smaller on average when the location parameter μ shifts from μ_0 down to $\mu_1(<\mu_0)$. Thus it is suitable for monitoring the changes in μ using $T_2(\mu_0,\lambda_0)=\overline{X}/\mu_0$.

3. Design of the control charts with the known parameters

In this section, we shall study how to design the ARL-unbiased control charts for λ and μ based on the monitoring statistics $T_1(\lambda_0)$ and $T_2(\mu_0,\lambda_0)$ when the in-control values λ_0 and μ_0 are known.

3.1. Design of control chart for the shape parameter

For a given false alarm rate α, let UCL_1 and LCL_1 be the upper and lower control limits of the control chart based on the monitoring statistic $T_1(\lambda_0)$, respectively. Then

$$P(LCL_1\leqslant T_1(\lambda_0)\leqslant UCL_1|\lambda=\lambda_0)=1-\alpha. \tag{2}$$

It is clear that there are infinite combinations of (LCL_1,UCL_1) to satisfy the Eq. (2). However, most of them are not ARL-unbiased. That is, a change of the process parameter might lead to an increased ARL, making it harder to be detected. Thus we need to look for a combination of (LCL_1,UCL_1) which can result in an ARL-unbiased control chart based on the monitoring statistic $T_1(\lambda_0)$. The following theorem provides the procedure to obtain the control limits of the ARL-unbiased control chart for λ.

Theorem 1. *For a given false alarm rate α, the upper and lower control limits of the ARL-unbiased control chart for λ, UCL_1 and LCL_1, satisfy the following equations*

$$\begin{cases}F_{\chi^2_{n-1}}(UCL_1)-F_{\chi^2_{n-1}}(LCL_1)=1-\alpha,\\ f_{\chi^2_{n+1}}(UCL_1)=f_{\chi^2_{n+1}}(LCL_1),\end{cases} \tag{3}$$

where $f_{\chi^2_k}(x)$ and $F_{\chi^2_k}(x)$ are the p.d.f. and the cumulative distribution function (c.d.f.) of the χ^2 distribution with k degrees of freedom, respectively.

Proof. Suppose that λ_0 and $\lambda_1=\rho\lambda_0$ are the in-control and true values of λ, respectively. Then the probability that a point falls within the control limits of the ARL-unbiased control chart is given by