ARTICLE IN PRESS

Mathematical and Computer Modelling ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at SciVerse ScienceDirect

Mathematical and Computer Modelling

journal homepage: www.elsevier.com/locate/mcm

Positive solutions of singular fractional differential equations with integral boundary conditions

SeakWeng Vong *

Department of Mathematics, University of Macau, Av. Padre Tomás Pereira Taipa, Macau, China

ARTICLE INFO

Article history:
Received 30 May 2011
Received in revised form 23 December 2011
Accepted 17 June 2012

Keywords:
Fractional differential equation
Positive solution
Singular problem
Fixed point theorem

ABSTRACT

In this paper, we study positive solutions of the nonlocal boundary value problem for a class of singular fractional differential equations. The existence of positive solutions is obtained by the method of upper and lower solutions together with the Schauder fixed point theorem.

1. Introduction

In recent years, the interest on the study of fractional differential equations has been growing rapidly. Applications of the fractional differential equations can be found in various areas, including engineering, physics and chemistry [1–3]. In this paper, we concentrate on the study of the positive solutions to the boundary value problems of fractional differential equations. Recently, there have been quite a lot of studies on these problems [4–15]. In the following, we give a brief review of those results that are closely related to this paper. To this end, we first recall the definition of fractional derivatives.

Definition 1.1. The Caputo fractional derivative of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is given by

$$^{c}D^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}}ds, \quad 0 < t < +\infty$$

where $n-1 < \alpha < n$, $n \in \mathbb{N}$, provided that the right-hand side is pointwisely defined.

In [16], the following boundary value problem of the singular fractional differential equation is considered

$$^{c}D^{\alpha}_{0+}u(t) + f(t, u(t)) = 0, \qquad u(0) = u'(1) = u''(0) = 0, \quad 0 < t < 1, \tag{1}$$

where $2 < \alpha \leq 3$ and $\lim_{t\to 0^+} f(t, \cdot) = \infty$. By the Krasnoselskii fixed point theorem and nonlinear alternative of Leray–Schauder type in a cone, Qiu and Bai establish the existence of positive solutions to the problem. Very recently, Wang et al. [17] studied (1) by exploiting the method of upper and lower solutions. They showed that the problem admits positive solutions when $f(t, u)$ satisfies some general conditions. In this paper, we follow the method proposed in [17] to study fractional differential equations with integral boundary conditions.

* Tel.: +853 8397 4359; fax: +853 28838314.
E-mail address: swvong@umac.mo.

0895-7177/$ – see front matter © 2012 Elsevier Ltd. All rights reserved.
doi:10.1016/j.mcm.2012.06.024

Fractional differential equations with integral boundary conditions have been studied in [18] using fixed point theory for cones and the Krein–Rutman theorem. In [18], the equation under consideration involves the Riemann–Liouville fractional derivative. Recall that the Riemann–Liouville fractional derivative is defined as

$$D_{0+}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{-\alpha+n-1}f(s)ds.$$

In general, the Caputo fractional derivative and the Riemann–Liouville fractional derivative of a function are not the same. In particular, the solution space of ${}^cD_{0+}^{\alpha}u(t) = 0$ is spanned by $\{1, t, \dots, t^{n-1}\}$, while the solution space of $D_{0+}^{\alpha}u(t) = 0$ is spanned by $\{t^{\alpha-1}, t^{\alpha-2}, \dots, t^{\alpha-n}\}$. Consequently, the Green functions to a problem corresponding to the two derivatives are different.

The Green function is one of the main tools for the study of differential equations. The arguments in [18] may not be applied directly to the problem when the Green function corresponding to the Riemann–Liouville fractional derivative is replaced by that for the Caputo fractional derivative. In Section 3, we adopt the method of upper and lower solution to study singular (Caputo) fractional differential equations with integral boundary conditions.

The remainder of this paper is organized as follows. Some basic concepts are reviewed in the next section. In Section 3, the existence of positive solutions to an integral boundary value problem for the fractional differential equations is established.

2. Preliminaries

Consider the fractional differential equation with an integral boundary condition:

$${}^cD_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, \qquad u'(0) = \cdots = u^{(n-1)}(0) = 0, \qquad u(1) = \int_0^1 u(s)d\mu(s), \tag{2}$$

where $n \geq 2$, $\alpha \in (n-1, n)$ and $\mu(s)$ is a function of bounded variation. Throughout this paper, we assume that f may have a singularity at $t = 1$ and

$$\int_0^1 d\mu(s) < 1. \tag{3}$$

The fractional integral is the main operator to study the fractional derivative, which is defined as:

Definition 2.1. The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function f is given by

$$I_{0+}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds,$$

provided that the right-hand side is pointwisely defined.

The following lemma shows that, under some assumptions, the fractional integrals are the inverse of the fractional derivatives.

Lemma 2.1. *If $f(t)$ is differentiable on $(0, 1)$ and $(1-t)^{\sigma}f(t) \in C[0, 1]$, where $\sigma \in [0, \alpha)$, $n-1 < \alpha < n$, then $I_{0+}^{\alpha}f$ is n-times differentiable on $(0, 1)$ and ${}^cD_{0+}^{\alpha}I_{0+}^{\alpha}f(t) = f(t)$.*

Proof. We note that $I_{0+}^{\alpha}f(t)$ is well-defined because

$$\int_0^t |(t-s)^{\alpha-1}f(s)|ds \leq M\int_0^t (t-s)^{\alpha-1}(1-s)^{-\sigma}ds \leq M\int_0^t (t-s)^{\alpha-1-\sigma}ds \leq \frac{M}{\alpha-\sigma}t^{\alpha-\sigma},$$

where $M = \sup_{t\in[0,1]}\{(1-t)^{\sigma}f(t)\}$.

For the differentiability, we first consider $\alpha \in (0, 1)$. Since $(1-t)^{\sigma}f(t) \in C[0, 1]$, one can see that $f(t) \in C[0, 1)$ and $f(0)$ is finite. Thus, we have

$$\begin{aligned}\frac{d}{dt}I_{0+}^{\alpha}f(t) &= \frac{-1}{\Gamma(\alpha+1)}\frac{d}{dt}\int_0^t f(s)d(t-s)^{\alpha}\\ &= \frac{1}{\Gamma(\alpha+1)}\frac{d}{dt}\left[f(0)t^{\alpha} + \int_0^t f'(s)(t-s)^{\alpha}ds\right]\\ &= \frac{1}{\Gamma(\alpha)}\left[f(0)t^{\alpha-1} + \int_0^t f'(s)(t-s)^{\alpha-1}ds\right],\end{aligned}$$