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Stability of a stochastic logistic model with distributed delay

Meng Liu [a,b,*], Ke Wang [b], Qiu Hong [b]

[a] School of Mathematical Science, Huaiyin Normal University, Huaian 223300, PR China
[b] Department of Mathematics, Harbin Institute of Technology, Weihai 264209, PR China

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ABSTRACT

This paper is concerned with the stability of the solutions to the stochastic logistic model with distributed delay, which is represented by the equation

$$dx(t) = x(t)\left(1 - ax(t) - b\int_{-\tau}^{0} x(t+\theta)d\mu(\theta)\right)[rdt + \sigma dB_t],$$

where B_t is a standard Brownian motion. This study shows that the above stochastic system has a global positive solution with probability 1 and establishes the sufficient conditions for stability of the zero solution and the positive equilibrium. Several numerical examples are introduced to illustrate the results. Some recent results are improved and generalized.

1. Introduction

Logistic equation is one of the most important and useful models in both ecology and mathematical ecology. The classical Logistic equation can be expressed as follows

$$dx(t)/dt = rx(t)\left(1 - ax(t)\right), \tag{1}$$

where $x(t)$ is the population size at time t, r denotes the growth rate, $a > 0$ represents the self-inhibition rate.

In Eq. (1) it is assumed that the growth of a population at any time t depends on the relative number of individuals at that time. In practice, "the current growth of a population should also be influenced by the past history of the species" (Gopalsamy [2]). Clearly, the size of the population at earlier times may affect the current resource availability. For example, if the size of a sheep population was large last year, the grassland may be destroyed. Restoration of grasslands will take lots of time. Then the current resource availability is reduced, which results in that the current growth of the population is reduced. Thus in order to describe the present effect of the past history of the species, several authors introduced the distributed delay into Eq. (1) and studied the following delayed Logistic equation

$$dx(t)/dt = rx(t)\left(1 - ax(t) - b\int_{-\tau}^{0} x(t+\theta)d\mu(\theta)\right), \tag{2}$$

where τ is a positive constant; $b > 0$ stands for the effect power of the past history; μ, a probability measure on $[-\tau, 0]$, is a weighting factor which indicates how much emphasis should be given to determine the present effect of the size of the population at earlier times on resource availability. There is an extensive literature concerned with the dynamics of Eq. (2)

* Corresponding author at: School of Mathematical Science, Huaiyin Normal University, Huaian 223300, PR China. Tel.: +86 051784183732.
E-mail address: liumeng0557@sina.com (M. Liu).

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and we here mention only Freedman and Wu [3], Kuang and Smith [4], Bereketoglu and Gyori [5], He and Gopalsamy [6], Faria [7] and Lisena [8] among many others. Particularly, the books by Kuang [1] and Gopalsamy [2] are good references in this area.

However, population systems in the real world are inevitably affected by the environmental noise. May [9] have pointed out that due to environmental noise, the birth rates in the system should exhibit random fluctuation. Consequently several authors introduced white noise into deterministic systems to reveal the effect of environmental noises on the population dynamics, see e.g. [10–27]. Particularly, Mao [11] has shown that the white noise can not only have a destabilising effect but can also have a stabilising effect in the control theory. These important results reveal the significant effect of white noise on the population systems. However, so far as we know a very little amount of work has been done with the stability of stochastic population systems with distributed delay.

Note that the parameter r denotes the growth rate, in practice, it is usually estimated by an average value plus a noise term. According to central limit theorem, the noise term follows a normal distribution. If we still use r to stand for the average growth rate, therefore we can replace r by $r + \sigma\dot{B}_t$, where $\dot{B}_t$ is a white noise and σ is a constant representing the intensity of the white noise. Then we obtain the following stochastic Logistic system with distributed delay:

$$dx(t) = x(t)\left(1 - ax(t) - b\int_{-\tau}^{0} x(t+\theta)d\mu(\theta)\right)\left[rdt + \sigma dB_t\right]. \tag{3}$$

System (3) includes many models which have been investigated by several researchers. For example, when $\sigma = 0$, the model was considered by Gopalsamy [2] and Kuang [1]. If $b = 0$, some modifications of the model (3) were investigated in [20–27]. Thus throughout this paper, we suppose that $\sigma \neq 0$ and $b > 0$.

Clearly Eq. (3) has two equilibria: 0 and $x^* = \frac{1}{a+b}$. The main aim of this paper is to investigate the stability of these equilibria. We shall show that

- For any initial data $\xi \in C([-\tau, 0], R_+)$, Eq. (3) has a unique and positive solution $x(t) = x(t; \xi)$ on $t \geq -\tau$, where $C([-\tau, 0], R_+)$ represents the family of continuous functions from $[-\tau, 0]$ to $R_+ = (0, +\infty)$ with the norm $\|\xi\| = \sup_{-\tau\leq\theta\leq 0} |\xi(\theta)|$.
- If $r < 0$ and $\xi(0) < 1/(a+b)$, then the zero solution of Eq. (3) is asymptotically stable a.s. (almost surely), i.e., the solution $x(t; \xi)$ obeys $\lim_{t\to+\infty} x(t; \xi) = 0$ a.s.
- If $a > b$ and $r > \frac{a^2+b^2}{a-b}x^*\sigma^2$, then the positive equilibrium x^* of Eq. (3) is globally asymptotically stable a.s., i.e., for any initial data $\xi \in C([-\tau, 0], R_+)$, the solution $x(t; \xi)$ satisfies $\lim_{t\to+\infty} x(t; \xi) = x^*$ a.s.

The rest of the paper is arranged as follows. In Section 2, we establish the existence-and-uniqueness theorem of the global positive solution for Eq. (3). In Section 3, we investigate the stability of the zero solution and positive equilibrium of Eq. (3). In Section 4, we work out some figures to illustrate our main results. The last section gives the conclusions.

2. Global positive solutions

Throughout this paper, let B_t be a standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$. To begin with, let us consider the following non-autonomous system

$$dx(t) = x(t)\left(1 - a(t)x(t) - b(t)\int_{-\tau}^{0} x(t+\theta)d\mu(\theta)\right)\left[r(t)dt + \sigma(t)dB_t\right], \tag{4}$$

where $a(t)$, $b(t)$, $r(t)$ and $\sigma(t)$ are continuous and bounded functions on $t \geq 0$, and $\inf_{t\geq 0} a(t) > 0$, $\inf_{t\geq 0} b(t) > 0$, $\inf_{t\geq 0} \sigma^2(t) > 0$. If $f(t)$ is a continuous and bounded function on $[0, +\infty)$, then define

$$\check{f} = \inf_{t\geq 0} f(t), \qquad \hat{f} = \sup_{t\geq 0} f(t).$$

Theorem 1. *For any initial data $\xi \in C([-\tau, 0], R_+)$, Eq. (4) has a unique and positive solution on $t \geq -\tau$.*

Proof. Note that the coefficients of Eq. (4) are locally Lipschitz continuous, then for any given initial value $\xi \in C([-\tau, 0], R_+)$, there is a unique maximal local solution $x(t)$ on $t \in [-\tau, \tau_e]$ to Eq. (4), where τ_e is the explosion time (see e.g. [28]). To prove this solution is global, we need only to demonstrate that $\tau_e = \infty$. Let $n_0 > 0$ be sufficiently large such that $\xi(0)$ lying within $[1/n_0, n_0]$. For each integer $n > n_0$, define the stopping times

$$\tau_n = \inf\{t \in [0, \tau_e] : x(t) \notin (1/n, n)\}.$$

It is easy to see that τ_n is increasing as $n \to \infty$. Let $\tau_\infty = \lim_{n\to+\infty} \tau_n$, then $\tau_\infty \leq \tau_e$ *a.s.* So we need only to show $\tau_\infty = \infty$. If this statement is false, there exists $\varepsilon \in (0, 1)$ such that $\mathcal{P}\{\tau_\infty < \infty\} > \varepsilon$. Consequently, there exists $T > 0$ and an integer $n_1 \geq n_0$ such that

$$\mathcal{P}\{\tau_n < T\} > \varepsilon, \quad n > n_1. \tag{5}$$