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Translation properties of time scales and almost periodic functions

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ABSTRACT

In this paper, we study the structure of time scales under translations. We define several kinds of time scales such as the two-way translation invariant time scale and investigate their properties. Then we develop the almost periodic function theory on time scales, with examples given along the way.

1. Introduction

The theory of time scales was first developed by Stefan Hilger [1] in order to unify continuous and discrete analysis. It has shown great potential for application recently, such as in the fields of economics [2,3], physics [4] and population dynamics [5,6]. Many important results have been obtained regarding time scales [7]. However, to the author's best knowledge, almost periodic function theories have not been extended to time scales. The goal of this paper is to study almost periodic functions on time scales.

The concept of almost periodic function was first studied by Harald Bohr [8] and later generalized by Abram Samoilovitch Besicovitch [9], Vyacheslav Stepanov and Hermann Weyl. It is useful in studying natural phenomenons that appear to retrace their paths, but not exactly, such as the planetary system. By studying the theory of almost periodic functions on time scales, we can broaden the fields of application for almost periodic functions. Almost periodic functions on $\mathbb{R}$ and almost periodic sequences on $\mathbb{Z}$ have been studied extensively, by developing almost periodic function theory on time scales, one can unify the two concepts and avoid proving results twice.

This paper begins the investigation by analyzing the structure of a time scale under translation. In Section 2, we first define translation invariants for the time scale $\mathbb{T}$ and the set of translation invariants $\Upsilon(\mathbb{T})$, then investigate the properties of $\Upsilon(\mathbb{T})$. Next, according to the structure of time scales, we define two-way translation invariant time scales, positive translation invariant time scales and negative translation invariant time scales, and investigate their properties respectively. Since $\tau \in \Upsilon(\mathbb{T})$ implies $n\tau \in \Upsilon(\mathbb{T})$ for $n \in \mathbb{N}^+$, we introduce the concept of primitive translation invariants. Section 3 introduces the concept of odd translation time scales, and proves that an odd translation time scale $\mathbb{T}$ either has no primitive translation invariants (in this case we have $\mathbb{T} = \mathbb{R}$), or has exactly one positive primitive translation invariant. Finally we get the important result that $\mathbb{T}$ is an odd translation time scale if and only if it is a two-way translation invariant time scale. Section 4 defines almost periodic functions and uniformly almost periodic functions on two-way translation invariant time scales and positive translation invariant time scales respectively, with examples given by theorems.

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2. Translation invariance of time scales

In order to define almost periodic functions on time scales, we need to investigate how time scales change under translations. We begin with the definition below.

Definition 1. For a given time scale $\mathbb{T}$, if a real number τ satisfies

$$t + \tau \in \mathbb{T}, \quad \text{for all } t \in \mathbb{T}, \tag{1}$$

then τ is called a translation invariant for $\mathbb{T}$.

We denote the set of all translation invariants of $\mathbb{T}$ as $\Upsilon(\mathbb{T})$.

The following properties are obvious.

(i) For a given time scale $\mathbb{T}$, if $\tau \in \Upsilon(\mathbb{T})$ then $n\tau \in \Upsilon(\mathbb{T})$ for $n \in \mathbb{N}^+$.
(ii) For any time scale $\mathbb{T}$, $0 \in \Upsilon(\mathbb{T})$.

We give several examples of $\Upsilon(\mathbb{T})$ below.

Example 1. Denote $[0, +\infty)$ as $\mathbb{R}^+$, $(-\infty, 0]$ as $\mathbb{R}^-$, then we can get

$$\Upsilon(\mathbb{R}) = \mathbb{R}, \qquad \Upsilon(\mathbb{Z}) = \mathbb{Z}, \qquad \Upsilon(\mathbb{R}^+) = \mathbb{R}^+, \qquad \Upsilon(\mathbb{R}^-) = \mathbb{R}^-.$$

$\Upsilon(\mathbb{T})$ may not be a subset of $\mathbb{T}$. See the example given below.

Example 2. Let

$$\mathbb{T} = \mathbb{Z} - 3\mathbb{Z} = \{\ldots, -5, -4, -2, -1, 1, 2, 4, 5, \ldots\}.$$

Apparently, $\Upsilon(\mathbb{T}) = 3\mathbb{Z}$, which is not a subset of $\mathbb{T}$.

For some time scales, $\Upsilon(\mathbb{T})$={0}.

Example 3. Let

$$\mathbb{T} = \{1, 4, 9, \ldots, n^2, \ldots\}.$$

Here we have $\Upsilon(\mathbb{T}) = \{0\}$.

Lemma 1. *$\Upsilon(\mathbb{T})$ is a closed set.*

Proof. Suppose τ is an element of the closure of $\Upsilon(\mathbb{T})$. Then there exists a real number sequence $\{\tau_i\}$ whose limit is τ, with $\tau_i \in \Upsilon(\mathbb{T})$. For a given $t \in \mathbb{T}$, we have $t + \tau_i \in \mathbb{T}$. It is easy to see that $t + \tau_i \to t + \tau$ as $i \to \infty$. According to the definition of time scales, $\mathbb{T}$ is a closed set, so we get $t + \tau \in \mathbb{T}$. Because t is arbitrary in $\mathbb{T}$, we obtain $\tau \in \Upsilon(\mathbb{T})$. The proof is complete. □

Since 0 is always an element of $\Upsilon(\mathbb{T})$, and it does not change the original time scale during the translation, we denote the set of nonzero translation invariants $\Upsilon(\mathbb{T}) - \{0\}$ as $\overline{\Upsilon}(\mathbb{T})$.

Definition 2. If $\overline{\Upsilon}(\mathbb{T}) \neq \emptyset$, and for all $\tau \in \overline{\Upsilon}(\mathbb{T})$, $\tau > 0$ is satisfied, then $\mathbb{T}$ is called positive translation invariant time scale.

If $\overline{\Upsilon}(\mathbb{T}) \neq \emptyset$, and for all $\tau \in \overline{\Upsilon}(\mathbb{T})$, $\tau < 0$ is satisfied, then $\mathbb{T}$ is called negative translation invariant time scale.

If there exist $\alpha, \beta \in \overline{\Upsilon}(\mathbb{T})$, with $\alpha > 0$, $\beta < 0$, then $\mathbb{T}$ is called two-way translation invariant time scale.

We have the following results.

Lemma 2. (i) *If $\mathbb{T}$ is a positive translation invariant time scale, then* $\sup \mathbb{T} = +\infty$.
(ii) *If $\mathbb{T}$ is a negative translation invariant time scale, then* $\inf \mathbb{T} = -\infty$.
(iii) *If $\mathbb{T}$ is a two-way translation invariant time scale, then* $\sup \mathbb{T} = +\infty$, $\inf \mathbb{T} = -\infty$.

Proof. Suppose $\mathbb{T}$ is a positive translation invariant time scale. Then there exists a $\tau_1 \in \Upsilon(\mathbb{T})$ with $\tau_1 > 0$ and a $t_1 \in \mathbb{T}$. So we get $t_1 + n\tau \in \mathbb{T}$, $n \in \mathbb{N}^+$. This implies $\sup \mathbb{T} = +\infty$.

This completes the proof of (i). The proofs of (ii) and (iii) are quite similar. □

Definition 3. Suppose $\alpha \neq 0$, $\alpha \in \Upsilon(\mathbb{T})$. We define

$$\mathbb{N}^+(\alpha) = \left\{ n \in \mathbb{N}^+ : \frac{\alpha}{n} \in \Upsilon(\mathbb{T}) \right\}.$$

It is easy to see that $1 \in \mathbb{N}^+(\alpha)$.