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On a class of starlike functions related to a shell-like curve connected with Fibonacci numbers

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ABSTRACT

In this paper we investigate an interesting subclass $\mathcal{SL}$ of analytic univalent functions in the open unit disc on the complex plane. This class was introduced by Sokół [J. Sokół, On starlike functions connected with Fibonacci numbers, Folia Scient. Univ. Tech. Resoviensis 175 (1999) 111–116]. The class $\mathcal{SL}$ is strongly related to the class $\mathcal{KSL}$ considered earlier by the authors of the present work in their paper [J. Dziok, R. K. Raina, J. Sokół, Certain results for a class of convex functions related to shell-like curve connected with Fibonacci Numbers, Comput. Math. Appl. 61 (2011) 2606–2613]. Apart from furnishing some genuine remarks, we present certain new results for the class $\mathcal{SL}$ of functions, and also mention some relevant cases for this function class.

1. Introduction

Let $\mathcal{A}$ be the class of all holomorphic functions f in the open unit disc Δ with the normalization $f(0) = 0, f'(0) = 1$, and let $\mathcal{S}$ denote the subset of $\mathcal{A}$ which is composed of univalent functions. We say that f is subordinate to F in Δ, written as $f \prec F$, if and only if $f(z) = F(\omega(z))$ for some holomorphic function ω such that $\omega(0) = 0$ and $|\omega(z)| < 1$, for all $z \in \Delta$. The class of starlike functions $\mathcal{S}^*$ can be defined in various ways, and for example, we say that $f \in \mathcal{A}$ is starlike if it satisfies the condition that

$$\frac{zf'(z)}{f(z)} \prec p(z) \quad (z \in \Delta), \tag{1.1}$$

where $p(z) = (1 + z)/(1 - z)$. Several subclasses of $\mathcal{S}^*$ have been defined in the literature by choosing appropriately the arbitrary function $p(z)$ in (1.1). We consider it worthwhile here to mention some useful geometric transformations that arise when the function $p(z)$ is chosen suitably. Thus, it is easily observed that when

(i) $p(z) = \frac{1+(1-2\alpha)z}{1-z}, \alpha < 1$, then under this transformation, the image of the unit circle $|z| = 1$ is a straight line $\Re(w) = \alpha$, while the image of the unit disc Δ is the half plane $\Re(w) > \alpha$. In this case, a function $f \in \mathcal{A}$ satisfying (1.1) is called starlike of order α and the family of all such functions is denoted by $\mathcal{S}^*(\alpha)$.

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(ii) $p(z)=\frac{1+Az}{1+Bz}$, $-1<B<A\le 1$, then $p(\Delta)$ is the disc $\mathbf{D}(C(A,B),R(A,B))$ with the centre $C(A,B)=(1+AB)/(1-B^2)$, and the radius $R(A,B)=(A+B)/(1-B^2)$; see [1,2].

(iii) $p(z)=\left(\frac{1+z}{1-z}\right)^{\beta}$, $0<\beta\le 1$, then $p(\Delta)$ is an angle $\{w\in\mathbf{C}:\operatorname{Arg}w<\beta\pi/2\}$; see [3]. In this case, a function $f\in\mathcal{A}$ satisfying (1.1), is called strongly starlike of order β.

(iv) $p(z)=1+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$, then after some elementary calculations, one can find that $p(\Delta)$ is a parabolic domain $\left\{w=u+iv\in\mathbf{C}:u>\sqrt{(u-1)^2+v^2}\right\}$. In this case, if a function $f\in\mathcal{A}$ satisfies (1.1), then the function $\int_0^z\frac{f(t)}{t}\,\mathrm{d}t$ is called a uniformly convex function; see [4–6].

(v) $p(z)=\frac{1}{1-\beta^2}\cos\left\{\frac{2}{\pi}(\arccos\beta)i\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right\}-\frac{\beta^2}{1-\beta^2}$, $0<\beta<1$, then $p(\Delta)$ is an interior of hyperbola $\left\{w=u+iv\in\mathbf{C}:u>\beta\sqrt{(u-1)^2+v^2}\right\}$; see [7–9].

(vi) $p(z)=1+\frac{2}{k^2-1}\sin^2\left(\frac{\pi}{2\mathcal{K}(t)}\mathcal{F}(\sqrt{z/t},t)\right)$, $k>1$, $\mathcal{F}(1,t)=\mathcal{K}(t)$, where

$$\mathcal{F}(w,t)=\int_0^w\frac{\mathrm{d}x}{\sqrt{1-x^2}\sqrt{1-t^2x^2}},$$

is called the Jacobi elliptic integral, and $t\in(0,1)$ is such that

$$k=\cosh\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)},$$

then $p(\Delta)$ is an elliptic domain $\left\{w=u+iv\in\mathbf{C}:u>\beta\sqrt{(u-1)^2+v^2}\right\}$; see [7–9].

(vii) $p(z)=\sqrt{1+z}$, where the branch of the square root is chosen in order that $\sqrt{1}=1$, then $p(\Delta)$ is an interior of the right loop of the Lemniscate of Bernoulli $\{w\in\mathbf{C}:\Re(w)>0,\ |w^2-1|<1\}$; see [10,11].

(viii) $p(z)=\left(\frac{1+z}{1+(1-b)/bz}\right)^{1/\alpha}$, $a\ge 1$, $b\ge 1/2$, where the branch of the square root is chosen in order that $p(0)=1$, then $p(\Delta)$ is a leaf-like domain $\{w\in\mathbf{C}:|w^{\alpha}-b|<b,\ \operatorname{Arg}w\le\pi/(2\alpha)\}$; see [12].

In cases (i)–(viii), the function p is a convex univalent function. In [13], Ma and Minda proved some general results for functions $f\in\mathcal{A}$ satisfying (1.1), where p is assumed to be univalent, $p(\Delta)$ is assumed to be symmetric with respect to real axis and starlike with respect to $p(0)=1$. The problems in the class defined by (1.1) become much more difficult if the function p is not univalent. We will consider such class of functions in the present work. An interesting case, when the function p is convex but is not univalent, was considered in [14]. It would be very interesting to find what extra information can be attained (or gained) by using the defining condition (1.1), instead of the weaker condition that

$$\frac{zf'(z)}{f(z)}\in p(\Delta),\quad\text{for all } z\in\Delta, \tag{1.2}$$

given a non-univalent p.

2. Main results

We first recall here the following class of functions introduced in [15], in which the estimates of coefficients and other connected results were investigated. The related classes of functions were also studied in [16,17].

Definition 1. The function $f\in\mathcal{A}$ belongs to the class $\mathcal{SL}$, if it satisfies the condition (1.1) with

$$\widetilde{p}(z)=\frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2}\quad(z\in\Delta), \tag{2.1}$$

where $\tau=(1-\sqrt{5})/2\approx -0.618$.

The function (2.1) is not univalent in Δ, but it is univalent in the disc $|z|<(3-\sqrt{5})/2\approx 0.38$. For example, $\widetilde{p}(0)=\widetilde{p}(-\frac{1}{2\tau})=1$ and $\widetilde{p}(e^{\pm i\arccos(1/4)})=\frac{\sqrt{5}}{5}$, and it may also be noticed that

$$\frac{1}{|\tau|}=\frac{|\tau|}{1-|\tau|},$$

which shows that the number $|\tau|$ divides $[0,1]$ such that it fulfils the golden section of this segment

Let us put $\Re\left[\widetilde{p}(e^{i\varphi})\right]=x$ and $\Im\left[\widetilde{p}(e^{i\varphi})\right]=y$, $\varphi\in[0,2\pi)\setminus\{\pi\}$; then upon performing simple calculations, we find that

$$x=\frac{\sqrt{5}}{2(3-2\cos\varphi)},\qquad y=\frac{\sin\varphi(4\cos\varphi-1)}{2(3-2\cos\varphi)(1+\cos\varphi)},\quad \varphi\in[0,2\pi)\setminus\{\pi\}.$$