Mathematical and Computer Modelling 57 (2013) 1212–1226

Contents lists available at SciVerse ScienceDirect

Mathematical and Computer Modelling

journal homepage: www.elsevier.com/locate/mcm

A viscosity scheme for mixed equilibrium problems, variational inequality problems and fixed point problems

Gang Cai *, Shangquan Bu

Department of Mathematical Sciences, Tsinghua University, 100084 Beijing, China

ARTICLE INFO

Article history:
Received 25 April 2011
Received in revised form 3 September 2012
Accepted 14 September 2012

Keywords:
Fixed point
Nonexpansive mapping
Variational inequality
General mixed equilibrium problem

ABSTRACT

In this paper, we study a new general iterative scheme for finding a common element of the set of solutions of finite general mixed equilibrium problems, the set of solutions of finite variational inequalities for cocoercive mappings, the set of solutions of common fixed points of an infinite family of nonexpansive mappings and the set of solutions of fixed points of a nonexpansive semigroup in Hilbert space. Strong convergence theorems are obtained under appropriate conditions.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. In what follows, we denote by "$\rightharpoonup$" weak convergence and by "$\to$" strong convergence. We denote by $F(T) = \{x \in C : Tx = x\}$ the set of fixed points of T, where T be a nonlinear mapping. We recall some definitions and notations adopted previously in [1,2].

Let $A : C \to H$ be a nonlinear mapping, $\varphi : C \to \mathbb{R}$ be a real-valued function and $F : C \times C \to \mathbb{R}$ be a bifunction. The generalized mixed equilibrium problem is to find $x \in C$ such that

$$F(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

We denote the set of solutions of (1.1) by $GMEP(F, \varphi, A)$.

When $\varphi = 0$, $A = 0$, problem (1.1) reduces to the following equilibrium problem which is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

We denote the set of solutions of (1.2) by $EP(F)$.

When $A = 0$, problem (1.1) reduces to the following mixed equilibrium problem (Ceng and Yao [3]) which is to find $x \in C$ such that

$$F(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.3}$$

We denote the set of solutions of (1.3) by $MEP(F, \varphi)$.

When $\varphi = 0$, problem (1.1) reduces to the following generalized equilibrium problem which is to find $x \in C$ such that

$$F(x, y) + \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.4}$$

We denote the set of solutions of (1.4) by $EP(F, A)$.

* Corresponding author.
E-mail addresses: caigang-aaaa@163.com (G. Cai), sbu@math.tsinghua.edu.cn (S. Bu).

0895-7177/$ – see front matter © 2012 Elsevier Ltd. All rights reserved.
doi:10.1016/j.mcm.2012.09.023

The problem (1.1) is more general because it includes fixed point problems, variational inequality problems, equilibrium problems, minimax problems as special cases and it has been studied extensively by many authors; see for instance, [4–19, 1,20–22].

A mapping $S : C \to C$ is called nonexpansive, if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

A mapping $A : C \to H$ is called monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

A mapping $A : C \to H$ is said to be relaxed (c, d)-cocoercive, if there exist two constants $c, d > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq -c\,\|Ax - Ay\|^2 + d\,\|x - y\|^2, \quad \forall x, y \in C.$$

A mapping $A : C \to H$ is called α-inverse strongly monotone if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\,\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

A mapping $f : C \to C$ is called a contraction if there exists a constant $\alpha \in (0, 1)$ such that

$$\|fx - fy\| \leq \alpha\,\|x - y\|, \ \forall x, y \in C.$$

Let G be an unbounded subset of $\mathbb{R}^+$, a family $\mathcal{W} := (W(s))_{s\in G}$ is called a nonexpansive semigroup on C if the following conditions hold:

(1) $W(0)x = x$ for all $x \in C$;
(2) $W(s + t)x = W(s)W(t)$ for all $s, t \in G$;
(3) for all $x \in C$, $s \to W(s)x$ is continuous;
(4) $\|W(s)x - W(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \in G$.

We denote by $F(\mathcal{W})$ the set of all common fixed points of $\mathcal{W}$. By Lemma 1 of [23], we have that $F(\mathcal{W})$ is closed and convex.

Recently, Chang et al. [18] introduced an iterative scheme by using the viscosity approximation method for finding a common element of the set of solutions of common fixed points for a family of infinitely nonexpansive mappings, the set of solutions of the variational inequality for an α-inverse-strongly monotone mapping and the set of solutions of an equilibrium problem (1.2) in Hilbert space. More precisely, they studied the following algorithm:

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\, y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n k_n, \quad \forall\, n \geq 1, \\ k_n = P_C(y_n - \lambda_n A y_n), \\ y_n = P_C(u_n - \lambda_n A u_n), \end{cases} \tag{1.5}$$

and proved some strong convergence under some suitable conditions.

Very recently, Peng and Yao [10] considered the following iterative algorithm for finding a common element of the set of solutions of problem (1.3), the set of solutions of common fixed points of a family of finitely nonexpansive mappings and the set of solutions of the variational inequality for a monotone, Lipschitz continuous mapping:

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \varphi(y) - \varphi(u_n) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\, y \in C, \\ y_n = P_C(u_n - \gamma_n A u_n), \\ x_{n+1} = \alpha_n v + \beta_n x_n + \gamma_n W_n P_C(u_n - \gamma_n A y_n), \quad \forall\, n \in \mathbb{N}. \end{cases} \tag{1.6}$$

The purpose of this paper is to introduce an iterative algorithm for finding a common element of the set of solutions of finite general mixed equilibrium problems, the set of solutions of common fixed points of an infinite family of nonexpansive mappings, the set of solutions of finite variational inequalities and the set of solutions of fixed points of a nonexpansive semigroup. Then we prove some strong convergence theorems for approximating a common element of the above four sets. The results obtained in this paper extend the corresponding results of Peng and Yao [10], Chang et al. [18] and many others.