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Some generalizations of Ekeland's variational principle with applications to fixed point theory

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ABSTRACT

In this paper, some extensions of the Ekeland variational principle in metric spaces are given for a generalized pseudodistance. As an application we obtain Caristi's fixed point theorem. Then, by using this result, we establish some fixed point theorems for set-valued contractive mappings.

1. Introduction and preliminaries

In 1972, Ekeland [1] proved an important theorem in nonlinear analysis. This theorem is called Ekeland's variational principle. It has significant applications in the geometry of Banach spaces, optimization theory, game theory, optimal control theory, dynamical systems, and others; see [1–6] and the references therein for more details. It is well known that the Ekeland variational principle is equivalent to many famous results, namely, the Caristi–Kirk fixed point theorem, the petal theorem, Phelp's lemma, Daneš' drop theorem, Oettli–Théra's theorem and Takahashi's theorem; see for example [3,7–14]. Recently, Kada et al. [15], Amemiya and Takahashi [16], Shioji et al. [17], Suzuki [18–20], Suzuki and Takahashi [21], Ansari [8], Al-Homidan and Ansari [9], Lin and Du [11], Khanh and Quy [22,23] improved and generalized Takahashi's nonconvex minimization theorem, Caristi's fixed point theorem and Ekeland's variational principle by using generalized distances: for example, w-distances, τ-distances, τ-functions, weak τ-functions, and Q-functions. Very recently, Benahmed and Azé [24], Uderzo [25], and Feng and Liu [26] introduced some notions of generalized nonlinear contractive mappings, and by using variational analysis proved some fixed point results for such mappings in complete metric spaces.

In this work, by using the concept of generalized pseudodistance, we obtain a version of Ekeland's variational principle. Moreover, we pose an equivalent problem with Kirk's problem [27–31], and we give a positive partial answer to this problem. Furthermore, some applications to the fixed point theory of generalized set-contractive mappings are given. Very recently, Ansari and Lin [3] provided a survey chapter that includes the results of [9,11,12]. They presented different forms of Ekeland's variational principle by using τ-functions in metric spaces and Q-functions in quasi-metric spaces. Here, we show that every τ-function and Q-function is a weak τ-function, and so they are generalized pseudodistances. However, by an example, we show that there is a generalized pseudodistance which is not a weak τ-function. Therefore, our results are different from the results in [3].

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Definition 1.1. Assume that (X, d) is a metric space. A function $w : X \times X \to \mathbb{R}_+$ is called a generalized pseudodistance on X if the following conditions hold.

$(w1)$ (triangle inequality) $w(x, z) \le w(x, y) + w(y, z)$, for all $x, y, z \in X$.
$(w2)$ For any sequence (x_n) in X such that

$$\lim_{n\to\infty} \sup_{m>n} w(x_n, x_m) = 0,$$

if there exists a sequence (y_n) in X satisfying

$$\lim_{n\to\infty} w(x_n, y_n) = 0,$$

then

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

The concept of generalized pseudodistance was first introduced and studied by Włodarczyk and Plebaniak [32].

Remark 1.2. If, in addition to $(w1)$ and $(w2)$, the following condition is also satisfied:

$(w3)$ $w(x, y) = 0$ and $w(x, z) = 0$ imply that $y = z$, for $x, y, z \in X$,

then the generalized pseudodistance is called a weak τ-function on X. This concept was introduced and studied by Khanh and Quy [22,23].

A weak τ-function on X is said to be a τ-function on X if the following condition holds.

$(w4)$ If $x \in X$ and $\{y_n\}$ in X with $\lim_{n\to\infty} y_n = y$ and $w(x, y_n) \le M$ for some $M = M(x) > 0$, then $w(x, y) \le M$.

This concept was first introduced and studied by Lin and Du [11], and they proved that every w-distance in the sense of Kada et al. [15] is a τ-function.

If (X, d) is a quasi-metric, then the function w is said to be a Q-function on X if conditions $(w1)$, $(w4)$, and the following condition are satisfied.

$(w5)$ For any $\varepsilon > 0$, there exists $\delta > 0$ such that $w(x, y) \le \delta$ and $w(x, z) \le \delta$ imply $d(x, z) \le \varepsilon$.

The notion of Q-function was first introduced and studied by Al-Homeidan et al. [9]. Also, recently such distances were studied by Ansari et al. [3]. Moreover, conditions $(w1)$ and $(w5)$ imply condition $(w2)$ by the proof of Remark 2.1 in [11]. It is clear that condition $(w3)$ can be obtained from condition $(w5)$. Therefore, every Q-function on a metric space is a weak τ-function.

Suzuki [18] introduced the concept of τ-distance and proved that every w-distance and Tataru distance is a τ-distance. Notice that it is known that the notions of a τ-function and a τ-distance are incomparable. Moreover, in [22], it is shown that any τ-distance is a weak τ-function. Several examples of w-distance, τ-distance, τ-function, and weak τ-function are given in [11,15,18,22].

Włodarczyk and Plebaniak [32] also proved that every τ-distance, τ-function, and Vályi distance is a generalized pseudodistance. Furthermore, they gave some examples of generalized pseudodistances which are neither a τ-distance, a τ- function nor a Vályi distance.

It is obvious that every weak τ-function is a generalized pseudodistance. But the following example shows that there exists a generalized pseudodistance which is not a weak τ-function.

Example 1.3. Let $w : [0, 2] \times [0, 2] \to [0, \infty)$ be defined as follows:

$$w(x, y) = \begin{cases} 0, & x - y = -2; \\ |x - y|, & -2 < x - y \le 0; \\ x - y + 2, & 0 < x - y \le 2. \end{cases}$$

We show that w is a generalized pseudodistance. Suppose that $x, y, z \in X$. We consider the following cases.

Case 1. If $x - y = -2$, then it is clear that $w(x, y) \le w(x, z) + w(z, y)$.

Case 2. Suppose that $-2 < x - y \le 0$. Let $x - z = -2$. Then $x = 0$ and $z = 2$. If $-2 \le z - y \le 0$, then $x - y \le -2$, which is impossible. If $0 < z - y \le 2$, then $w(x, y) = |y| = y \le 2 - y + 2 = w(x, z) + w(z, y)$.

Let $-2 < x - z \le 0$. If $-2 < z - y \le 0$, then it is clear that $w(x, y) \le w(x, z) + w(z, y)$. If $z - y = -2$, then $x - y \le -2$, which is impossible. If $0 < z - y \le 2$, then $z - y > -1$, and so $y - x \le z - x + z - y + 2$; that is, $w(x, y) \le w(x, z) + w(z, y)$.

Let $0 < x - z \le 2$. If $z - y = -2$, then $x = 0$ and $z = 2$, and so $w(x, y) = |y| = y \le 2 - y + 2 = w(x, z) + w(z, y)$. If $-2 < z - y \le 0$, then it is clear that $w(x, y) \le w(x, z) + w(z, y)$. If $0 < z - y \le 2$, then $x - y > 0$, which is impossible.

Case 3. Suppose that $0 < x - y \le 2$. Let $x - z = -2$. Then $x = 0$ and $z = 2$. Therefore, $x - y = -y \le 0$, which is impossible.

Let $0 < x - z \le 2$. If $z - y = -2$, then $-2 < x - y \le 0$, which is impossible. If $-2 < z - y \le 0$, then it is clear that $w(x, y) \le w(x, z) + w(z, y)$. If $0 < z - y \le 2$, then it is clear that $w(x, y) \le w(x, z) + w(z, y)$.