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The general iterative scheme for semigroups of nonexpansive mappings and variational inequalities with applications

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ABSTRACT

In this paper, we introduce the implicit and explicit viscosity iteration schemes for nonexpansive semigroups $\{T(t) : t \geq 0\}$. Additionally, it proves that the proposed iterative schemes converge strongly to a unique common fixed point of $\{T(t) : t \geq 0\}$ in the framework of reflexive and strictly convex Banach space, which solves some variational inequality. The main results of this paper improve and generalize recent known results in the current literature.

1. Introduction

Let E be a real Banach space, K be a nonempty closed convex subset of E. Recall that a mapping $f : K \to K$ is a contraction on K if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|,\ x, y \in K$. We use Π_K to denote the collection of mappings f verifying the above inequality. That is, $\Pi_K = \{f : K \to K \mid f$ is a contraction with constant $\alpha\}$. Note that each $f \in \Pi_K$ has a unique fixed point in K.

Recall that a one-parameter family $\mathscr{T} = \{T(t) : 0 \leq t < \infty\}$ is said to be a Lipschitzian semigroup on K (see, e.g., [1]) if the following conditions are satisfied.

(i) $T(0)x = x, x \in K$.
(ii) $T(t + s)x = T(t)T(s)x,\ t, s \geq 0,\ x \in K$.
(iii) For each $x \in K$, the map $t \mapsto T(t)x$ is continuous on $[0, \infty)$.
(iv) There exists a bounded measurable function $L_t : (0, \infty) \to (0, \infty)$ such that, for each $t > 0$, $\|T(t)x - T(t)y\| \leq L_t \|x - y\|,\ x, y \in K$.

A Lipschitzian semigroup $\mathscr{T}$ is said to be nonexpansive semigroup if $L_t = 1$ for all $t > 0$, and asymptotically nonexpansive if $\limsup_{t\to\infty} L_t \leq 1$, respectively. We use $F(\mathscr{T})$ to denote the common fixed point set of the semigroup $\mathscr{T}$, that is, $F(\mathscr{T}) = \{x \in K : T(s)x = x,\ \forall s > 0\}$.

A continuous operator of the semigroup $\mathscr{T}$ is said to be uniformly asymptotically regular (u.a.r.) on K if for all $h \geq 0$ and any bounded subset C of K, $\lim_{t\to\infty} \sup_{x\in C} \|T(h)T(t)x - T(t)x\| = 0$ (see [2]).

The variational inequality problem was first introduced by Hartman and Stampacchia [3]. Then, the variational inequality has achieved an increasing attention in many research fields, such as mathematical programming, constrained linear and nonlinear optimization, automatic control, manufacturing system design, signal and image processing and the complementarity problem in economics and pattern recognition (see [4–6] and the references therein). Nowadays, the

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theory of variational inequalities and fixed point theory are two important and dynamic areas in nonlinear analysis and optimization. One promising approach to handle these problems is to develop some kind of iterative schemes to compute the approximate solutions of variational inequalities and to find a common fixed point of a given family of operators. We now have a variety of techniques to suggest and analyze various iterative algorithms for solving variational inequalities and the related optimization problems. The fixed point theory has played an important role in the development of various algorithms for solving variational inequalities.

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Assume that A is strongly positive bounded linear operator on H, that is, there is a constant $\bar{\gamma} > 0$ with property

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2 \quad \forall x \in H.$$

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle,$$

where C is the fixed point set of a nonexpansive mapping T on H and b is a given point in H.

Recently, Marino and Xu [7] have considered the following general iteration process

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \tag{1.1}$$

and proved that if the sequence $\{\alpha_n\}$ satisfies appropriate conditions, the sequence $\{x_n\}$ generated by (1.1) converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad x \in \mathrm{Fix}(T),$$

which is the optimality condition for the minimization problem

$$\min_{x \in \mathrm{Fix}(T)} \frac{1}{2} \langle Ax, x \rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$, for $x \in H$) and $\mathrm{Fix}(T) = \{x \in H : Tx = x\}$.

Question. Can the iteration sequence (1.1) provide the same result for the more general class of continuous semigroup of nonexpansive mappings in Banach space?

Inspired by the above results, the purpose of this paper is to study the convergence problems of the implicit and explicit viscosity iterative processes for nonexpansive semigroup $\{T(t) : t \geq 0\}$ in general Banach spaces. And it provides an affirmative answer to the above Question. We establish the strong convergence results which generalize the corresponding results given by Marino and Xu [7], and Song and Xu [8].

2. Preliminaries

Recall that a Banach space E is said to be strictly convex if $\|x\| = \|y\| = 1,\ x \neq y$ implies $\|x + y\|/2 < 1$. In a strictly convex Banach space E, we have if $\|x\| = \|y\| = \|tx + (1 - t)y\|$, for $t \in (0, 1)$ and $x, y \in E$, then $x = y$.

Let E be a Banach space with dimension $E \geq 2$. The modulus of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\varepsilon) = \inf\{\, 1 - \|x + y\|/2 : \ \|x\| = \|y\| = 1,\ \|x - y\| = \varepsilon \,\}.$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.

Let $S(E) = \{\, x \in E : \|x\| = 1 \,\}$. The space E is said to be smooth if

$$\lim_{t \to 0} (\|x + ty\| - \|x\|)/t$$

exists for all $x, y \in S(E)$. For any $x, y \in E (x \neq 0)$, we denote this limit by (x, y). The norm $\| \cdot \|$ of E is said to be Fréchet differentiable if for all $x \in S(E)$, the limit (x, y) exists uniformly for all $y \in S(E)$. E is said to have a uniformly Gâteaux differentiable norm if for each $y \in S(E)$ the limit (x, y) is attained uniformly for $x \in S(E)$.

Let E^* denote the dual space of a Banach space E. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{x^* \in E^* : \ \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized pairing. If E is a Hilbert space, then $J = I$ (the identity mapping). It is well-known that if E is smooth, then J is single-valued, which is denoted by j. And if E has a uniformly Gâteaux differentiable norm then the duality mapping is norm-to-weak* uniformly continuous on bounded sets.

Let μ be a continuous linear functional on l^∞ and $(a_0, a_1, \ldots) \in l^\infty$. We write $\mu_n(a_n)$ instead of $\mu((a_0, a_1, \ldots))$. Recall a Banach limit μ is a bounded function functional on l^∞ such that

$$\|\mu\| = \mu_n(1) = 1, \quad \liminf_{n \to \infty} a_n \leq \mu_n(a_n) \leq \limsup_{n \to \infty} a_n,$$

$$\mu_n(a_{n+r}) = \mu_n(a_n)$$

for any fixed positive integer r and for all $(a_0, a_1, \ldots) \in l^\infty$.