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Further improvements of some double inequalities for bounding the gamma function

Cristinel Mortici *

Valahia University of Târgovişte, Department of Mathematics, Bd. Unirii 18, 130082 Târgovişte, Romania

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ABSTRACT

The aim of this paper is to establish some inequalities for bounding the gamma function that yields sharp asymptotic estimates for $\Gamma(x)$ as x tends to ∞. Our results improve several known results stated by Alzer [H. Alzer, Inequalities for the gamma function, Proc. Amer. Math. Soc., 128 (1) (1999) 141–147], Andreson and Qiu [G.D. Anderson, S.-L. Qiu, A monotonicity property of the gamma function, Proc. Amer. Math. Soc., 125 (1997), 3355–3362], and Li and Chen [J. Choi, Some mathematical constants, Appl. Math. Comput., 187 (2007) 122–140].

1. Introduction and motivation

In the past, especially in recent decades, several authors proved many interesting inequalities for the Euler gamma function (we refer the reader to [1–21] and all the references given therein). The aim of this article is to present new inequalities which improve some results given by Alzer [22], Anderson and Qiu [23], and Li and Chen [24].

Anderson and Qiu [23] used the increasing monotonicity of the function

$$x \mapsto \frac{\ln \Gamma(x+1)}{x \ln x}, \quad x > 1$$

to prove that for every $x > 1$,

$$x^{(1-\gamma)x-1} < \Gamma(x) < x^{x-1}, \tag{1}$$

where $\gamma = 0.577215\cdots$ is the Euler–Mascheroni constant.

The inequalities of type (1) have attracted the attention of many researchers, because of their simple form, and of their usefulness in practical applications in pure mathematics or other branches of science such as probabilities, engineering, or statistical physics.

Alzer [22, Theorem 2] refined and sharpened (1), proving that

$$x^{\alpha(x-1)-\gamma} < \Gamma(x) < x^{\beta(x-1)-\gamma}, \quad x > 1, \tag{2}$$

where the constants $\alpha = \left(\pi^2/6 - \gamma\right)/2$ and $\beta = 1$ are the best possible.

A simple calculation shows that for all sufficiently large x the bounds given in (2) for $\beta = 1$ are better than those corresponding to α. As a consequence, performant approximations of the form

$$\Gamma(x) \approx x^{\beta(x)(x-1)-\gamma}, \quad x \to \infty$$

* Tel.: +40 722727627.
E-mail addresses: cristinelmortici@yahoo.com, cmortici@valahia.ro.

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are obtained if and only if $\beta(x)$ becomes closer to 1, as x approaches infinity. Motivated by this remark, we improve (2), by proving the following double inequality

$$x^{\beta_1(x)(x-1)-\gamma} < \Gamma(x) < x^{\beta_2(x)(x-1)-\gamma}, \quad x \ge 55, \tag{3}$$

where

$$\beta_1(x) = 1 - \frac{1}{\ln x} + \left(\gamma + \frac{1}{2}\right)\frac{1}{x} - \frac{\mu}{x\ln x}$$

with $\mu = \ln \frac{e}{\sqrt{2\pi}} = 0.081061\ldots$, and

$$\beta_2(x) = 1 - \frac{1}{\ln x} + \left(\gamma + \frac{1}{2}\right)\frac{1}{x}.$$

Another direction for refinement (1) was introduced by Li and Chen [24], who proved that

$$\frac{x^{x-\gamma}}{e^{x-1}} < \Gamma(x) < \frac{x^{x-1/2}}{e^{x-1}}, \quad x \ge 3, \tag{4}$$

where the constants γ and $1/2$ are the best possible. It is to be noticed that for all sufficiently large x the upper bound given in (4) is better than the lower bound. As a consequence, performant approximations of the form

$$\Gamma(x) \approx \frac{x^{x-\delta(x)}}{e^{x-1}}, \quad x \to \infty$$

are obtained if and only $\delta(x)$ becomes closer to $1/2$, as x approaches infinity. Motivated by this remark, we improve (4), by proving the following double inequality

$$\frac{x^{x-\delta_1(x)}}{e^{x-1}} < \Gamma(x) < \frac{x^{x-\delta_2(x)}}{e^{x-1}}, \quad x > 1, \tag{5}$$

where

$$\delta_1(x) = \frac{1}{2} + \frac{\mu}{\ln x}, \qquad \delta_2(x) = \frac{1}{2} + \frac{\mu}{\ln x} - \frac{1}{12x}.$$

The results (1), (2) and (4) were stated using monotonicity and convexity properties of some functions which are connected with the gamma and psi functions and their derivatives.

We mainly use here the following inequality, for every $x > 0$,

$$\ln\sqrt{2\pi} + \frac{1}{2}\ln x + x\ln x - x + \frac{1}{12x} - \frac{1}{360x^3} < \ln\Gamma(x+1) < \ln\sqrt{2\pi} + \frac{1}{2}\ln x + x\ln x - x + \frac{1}{12x}. \tag{6}$$

This double inequality is a consequence of an excellent result of Alzer [25, Theorem 8], also mentioned in [22, Relation 4.7].

Our method is quite elementary and we are convinced that it is suitable for obtaining other new, performant estimates for the gamma function and for the special functions in general.

2. The results

Theorem 1. *For every $x \ge 55$, we have:*

$$x^{\left(1-\frac{1}{\ln x}+\left(\gamma+\frac{1}{2}\right)\frac{1}{x}-\frac{\mu}{x\ln x}\right)(x-1)-\gamma} < \Gamma(x) < x^{\left(1-\frac{1}{\ln x}+\left(\gamma+\frac{1}{2}\right)\frac{1}{x}\right)(x-1)-\gamma} \tag{7}$$

(the left-hand side inequality holds for every $x \ge 2$).

Proof. By taking the logarithm and replacing x by $x+1$, the left-hand side inequality (7) can be written as

$$\ln\Gamma(x+1) > x\ln(x+1) - x + \left(\gamma + \frac{1}{2}\right)\frac{x\ln(x+1)}{x+1} - \frac{\mu x}{x+1} - \gamma\ln(x+1).$$

Multiplying by $x+1$ and using (6), it suffices to show that $f(x) > 0$, where

$$f(x) = (x+1)\left(\ln\sqrt{2\pi} + \frac{1}{2}\ln x + x\ln x - x + \frac{1}{12x} - \frac{1}{360x^3}\right) - x(x+1)\ln(x+1) + x(x+1)$$
$$- \left(\gamma + \frac{1}{2}\right)x\ln(x+1) + \mu x + \gamma(x+1)\ln(x+1).$$