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Critical exponents and Pohozhaev identities of some partial differential equations

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ABSTRACT

By adopting the theory of Lie symmetry and Noether theorem, we investigate critical exponents and Pohozhaev identities of partial differential equation systems which was brought up to address the problem of finding the best constant in the weighted Hardy–Littlewood–Sobolev inequality.

1. Introduction

Suppose Ω is a star-shaped domain. Pohozhaev proved in [1] that solutions of Dirichlet problem

$$\begin{cases} -\Delta u(x) = |u|^{p-1}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

must satisfy the following identity, which is known as the Pohozhaev identity,

$$-\frac{1}{2}\int_{\partial\Omega} |Du|^2 (x\cdot\nu)ds = \int_{\Omega}\left(\frac{n-2}{2} - \frac{n}{p+1}\right)|u|^{p+1}dx. \tag{1.2}$$

By virtue of this Pohozhaev identity, one can study the nonexistence of positive solutions of Eq. (1.1). Specifically speaking, identity (1.2) implies that in super-critical case $\left(p > \frac{2n}{n-2} - 1\right)$ Eq. (1.1) has no positive solutions in star-shaped domain. In its critical $\left(p = \frac{2n}{n-2} - 1\right)$ and sub-critical $\left(p < \frac{2n}{n-2} - 1\right)$ cases, one may use constrained minimization to obtain a nontrivial solution.

Chen and Li [2], Li and Zhu [3] have studied this Dirichlet problem in R^n, that is,

$$\Delta u(x) = |u|^{p-1}u, \quad x \in R^n. \tag{1.3}$$

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They proved respectively that in sub-critical case $\left(p < \frac{2n}{n-2} - 1\right)$, Eq. (1.3) has no positive solution; and in the critical case $\left(p = \frac{2n}{n-2} - 1\right)$, its positive solution $u(x)$ must be symmetric with the form

$$u(x) \equiv \left(\frac{a}{1 + a^2|x - \bar{x}|^2}\right)^{\frac{n-2}{2}},$$

where $a > 0$ and $\bar{x} \in R^n$.

For this reason, the exponent $p^* = \frac{2n}{n-2} - 1$ is called the critical exponent for the Laplace operator. It corresponds to the loss of compactness of the continuous Sobolev embedding $H_0^1(\Omega) \subset L^q(\Omega)$ which is compact only when $1 \le q < 2^* = \frac{2n}{n-2}$.

The celebrated critical exponent and Pohozhaev identity [1,4] are important tools in the theory of partial differential equations. Among a variety of their applications, it is particularly useful in establishing nonexistence results. Recently, Bozhkov and Mitidieri [5] have proposed a general unified method to generate critical exponent and Pohozhaev identity. This approach is based on Noether identity and Lie symmetry theory, which will be introduced in Section 2.

In this paper, we investigate critical exponent and Pohozhaev identity of the following system of semilinear partial differential equations

$$\begin{cases} -\Delta(|x|^\alpha u(x)) = \dfrac{v^p(x)}{|x|^\beta}, & u > 0, \\ -\Delta(|x|^\beta v(x)) = \dfrac{u^q(x)}{|x|^\alpha}, & v > 0, \end{cases} \tag{1.4}$$

where α, β, p, q are nonnegative constants. This system is closely related to integral system

$$\begin{cases} u(x) = \dfrac{1}{|x|^\alpha} \displaystyle\int_{R^n} \dfrac{v^p(y)}{|y|^\beta |x-y|^{n-2}} dy, \\ v(x) = \dfrac{1}{|x|^\beta} \displaystyle\int_{R^n} \dfrac{u^q(y)}{|y|^\alpha |x-y|^{n-2}} dy, \end{cases} \tag{1.5}$$

which is the Euler–Lagrange equations of the problem investigated by Lieb [6] of finding the best constant in the well known weighted Hardy–Littlewood–Sobolev inequality [7],

$$J(f,g) = \int_{R^n}\int_{R^n} \frac{f(x)g(y)}{|x|^\alpha |x-y|^{n-2}|y|^\beta} dxdy \le C(n, r, s, \alpha, \beta)\|f\|_r \|g\|_s \tag{1.6}$$

where $1 < r, s < \infty, \alpha + \beta \ge 0, \frac{1}{r} + \frac{1}{s} + \frac{n-2+\alpha+\beta}{n} = 2, \frac{2}{n} - \frac{1}{r} < \frac{\alpha}{n} < 1 - \frac{1}{r}$.

Recently, Jin and Li [8] established the radial symmetry of positive solutions of Eq. (1.5). Shortly afterwards, Li and Lim [9] demonstrated that its positive solutions are asymptotic to certain forms around the center and near infinity respectively.

It is well known that when $\alpha = \beta = 0$, (1.4) is simplified to be the Lane–Emden equation system,

$$\begin{cases} -\Delta u(x) = v^p(x), & u > 0, \\ -\Delta v(x) = u^q(x), & v > 0. \end{cases} \tag{1.7}$$

The kind of Lane–Emden equation was widely studied by many researchers, such as Farina [10], Grumiau and Troestler [11]. Recently, we have obtained in [12] the existence and symmetry of positive solutions of the integral form of System (1.7) via the method of moving planes and the method of moving spheres.

It was proved by Chen and Li [13] that in the critical case $p = q = \frac{n+2}{n-2}$ system (1.7) becomes a single equation

$$-\Delta u(x) = u^{\frac{n+2}{n-2}}(x), u(x) > 0, \quad x \in R^n. \tag{1.8}$$

This equation is related to the well-known Yamabe problem, the prescribing scalar curvature problem, et al. This kind of problem has been studied by Chang and Yang [14], Li and Zhu [3], Chen and Li [15], et al..

The goal in this paper is to give a critical exponent for some common partial differential equations via the Lie symmetry theory. Our main results are as follows.

Theorem 1.1. (i) *The critical exponent of System (1.4) is*

$$\frac{n}{p+1} + \frac{n}{q+1} = n - 2 + \alpha + \beta.$$

(ii) *Suppose $u = v = 0$ on $\partial\Omega$. Then Pohozhaev identity of System (1.4) in a star-shaped domain is*

$$\int_\Omega \left(a + \frac{n}{q+1}\right) u^{q+1} + \left(b + \frac{n}{p+1}\right) v^{p+1} = \int_{\partial\Omega} |x|^{\alpha+\beta} \frac{\partial u}{\partial \nu}\frac{\partial v}{\partial \nu}(x, \nu) ds,$$

where a and b are real numbers satisfying that

$$a + b = 2 - n - \alpha - \beta$$

and ν is the outward unit normal to $\partial\Omega$.