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Asymptotic properties of iterates of certain positive linear operators

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ABSTRACT

In this paper we prove Korovkin type theorem for iterates of general positive linear operators $T : C[0, 1] \to C[0, 1]$ and derive quantitative estimates in terms of modulus of smoothness. In particular, we show that under some natural conditions the iterates $T^m : C[0, 1] \to C[0, 1]$ converges strongly to a fixed point of the original operator T. The results can be applied to several well-known operators; we present here the q-MKZ operators, the q-Stancu operators, the genuine q-Bernstein–Durrmeyer operators and the Cesaro operators.

1. Introduction

These iterated Bernstein operators were investigated in the 60's and 70's by P.C. Sikkema [1], R.P. Kelisky & T.J. Rivlin [2], S. Karlin & Z. Ziegler [3], J. Nagel [4], M.R. da Silva [5] and Gonska [6,7]. Some of this research was later generalized by Altomare et al. (see, for example, [8–10]). Altomare suggested to use in this context an approach described by Dickmeis and Nessel [11]. This was done recently by Rasa in [12,13]. Other new papers related to the subject of this article was written by S. Ostrovska [14] on iterates of q-Bernstein polynomials and N.I. Mahmudov [15] on iterates of positive linear operators which preserves e_2.

The methods employed to study the convergence of iterates of some operators occurring in the Approximation Theory include Matrix Theory methods, like stochastic matrices [16–18], Korovkin-type theorems [3], quantitative results about the approximation of functions by positive linear operators [19,20], fixed point theorems [21–23], or methods from the theory of C_0-semigroups, like Trotter's approximation theorem [3,24]. However, these techniques fail to work for the Meyer-Kőnig and Zeller (MKZ) or the May operators. Very recently, I. Gavrea and M. Ivan [25] proved that the iterates of the MKZ operates converges strongly to $P(f; x) = (1-x)f(0) + xf(1)$. Once such convergence have been obtained, the following natural question is to ask for rates of convergence. In Section 3, as a consequence of our results, we obtain the quantitative estimates for the iterates of the q-MKZ ($0 < q \le 1$) operators, which is completely new.

On the other hand, because of its powerful applications, Korovkin's result has been extended in many directions. There is an extensive literature on Korovkin-type theorems, which may have had a summit already about twenty five years ago. In particular, there exist abstract results that cover many naturally arising concrete cases. The contributions up to about 1994

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are excellently documented in the book of Altomare and Campiti [10]. More recent results obtained in [26,27,15] cover also approximation of q-type operators.

In this paper we establish quantitative Korovkin type theorem for the iterates of certain positive linear operators $T : C[0,1] \to C[0,1]$. Notice that, the special case when T preserves e_2, that is, when $T(e_2; x) = x^2$, $0 \le x \le 1$ was considered in [15] and results of [15] include only King type q-operators. In this paper, as a consequence of our main results, we obtain the quantitative estimates for the iterates of almost all classical and new positive linear operators: the q-MKZ operators, the q-Stancu operators, the genuine q-Bernstein–Durrmeyer operators in the case $0 < q \le 1$ and the Cesaro operators. It is worth mentioning that for $q = 1$ these operators become classical MKZ, Stancu and genuine Bernstein–Durrmeyer operators.

2. Main results

The following notations will be used throughout this paper. The classical Petree's K-functional and the second modulus of smoothness of a function f are defined respectively by

$$K_2(f,t) := \inf_{g \in C^2[0,1]} \left\{ \|f-g\| + t \left\|g''\right\| \right\}$$

and

$$\omega_2(f,t) := \sup_{0<h\le t} \sup_{0\le x\le 1-2h} |f(x+2h) - 2f(x+h) + f(x)|.$$

It is known that there exists a constant $C > 0$ such that

$$K_2(f,t) \le C\omega_2(f,\sqrt{t}). \tag{1}$$

Let $e_i : [0,1] \to R$ be the monomial functions $e_i(x) = x^i$, $i = 0, 1, 2$.

Now we formulate the main results of the paper. First result shows that under the conditions (2) the iterates of $T : C[0,1] \to C[0,1]$ converge to some linear positive operator $T^\infty : C[0,1] \to C[0,1]$.

Theorem 1. *Suppose that $T : C[0,1] \to C[0,1]$ is a positive linear operator such that*

$$\begin{aligned} &T(e_0) = e_0, \qquad T(e_1; x) \le x, \\ &\lim_{m\to\infty} \left\|T^m(e_1) - f_1\right\| = \lim_{m\to\infty} \left\|T^m(e_2) - f_2\right\| = 0, \quad f_1, f_2 \in C[0,1]. \end{aligned} \tag{2}$$

Then there exists a linear positive operator $T^\infty : C[0,1] \to C[0,1]$ such that the following pointwise estimate

$$\left|\left(T^m - T^\infty\right)(f;x)\right| \le k\omega_2\left(f, \sqrt{\lambda_m(x)}\right) + k\|f\|\,\delta_m(x) \tag{3}$$

holds true for $x \in [0,1]$ and $f \in C[0,1]$, where k is an absolute constant and

$$\begin{aligned} &\lambda_m(x) = \max\left\{\left|\left(T^m - T^\infty\right)(e_1;x)\right|, \left|\left(T^\infty - T^m\right)(e_2;x)\right|\right\}, \\ &\delta_m(x) = \left|\left(T^m - T^\infty\right)(e_1;x)\right|. \end{aligned}$$

Proof. It follows from the proof of [15, Theorem 1] that

$$\left|\left(T^{m+p} - T^m\right)(g;x)\right| \le \frac{1}{2}\left\|g''\right\| \left|\left(T^{m+p} - T^m\right)(e_2;x)\right| + \left(\left\|g''\right\| + \left\|g'\right\|\right)\left|\left(T^m - T^{m+p}\right)(e_1;x)\right|. \tag{4}$$

So $\{T^m(f;x)\}$ is a Cauchy sequence in $C[0,1]$ and there is a linear positive operator $T^\infty(f)$ such that

$$\lim_{m\to\infty} \left\|T^m(f) - T^\infty(f)\right\| = 0$$

for any $f \in C[0,1]$. Taking the limit as $p \to \infty$ in (4) and using the well known inequality

$$\left\|g'\right\| \le C_1\left(\|g\| + \left\|g''\right\|\right)$$

we have

$$\begin{aligned} \left|\left(T^\infty - T^m\right)(g;x)\right| &\le \frac{1}{2}\left\|g''\right\| \left|\left(T^\infty - T^m\right)(e_2;x)\right| + \left(\left\|g''\right\| + \left\|g'\right\|\right)\left|\left(T^m - T^\infty\right)(e_1;x)\right| \\ &\le \left(\frac{3}{2} + C_1\right)\lambda_m(x)\left\|g''\right\| + C_1\delta_m(x)\|g\|. \end{aligned} \tag{5}$$