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Empirical Bayesian regularization of the inverse acoustic problem

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ABSTRACT

This paper answers the challenge as how to automatically select a good regularization parameter when solving inverse problems in acoustics. A Bayesian solution is proposed that consists either in directly finding the most probable value of the regularization parameter or, indirectly, in estimating it as the ratio of the most probable values of the noise and source expected energies. It enjoys several nice properties such as ease of implementation and low computational complexity (the proposed algorithm boils down to searching for the global minimum of a 1D cost function). Among other advantages of the Bayesian approach, it makes possible to appraise the sensitivity of the reconstructed acoustical quantities of interest with respect to regularization, a performance that would be otherwise arduous to achieve.

1. Introduction

The inverse acoustic problem aims at reconstructing an acoustic quantity of interest (e.g. parietal pressure, particle velocity, acoustical intensity) from a limited number of remote measurements – as typically returned by an array of microphones or probes. As well-known, this is an ill-posed problem in the sense of Hadamard – i.e. it may have no solution at all or the solution may not be unique and it may be extremely sensitive to slight changes in the initial conditions [1] – for its exact solution would require measuring the complete field over a surface enclosing the source(s) of interest. As a consequence, solutions of an ill-posed inverse acoustic problems are typically found unstable with respect to very small perturbations in the measurements. Since ill-posedness fundamentally results from unavoidable loss of information during the measurement process, the usual cure is to *regularize* – i.e. to modify – the inverse operator so as to control the magnitude or energy of the expected solution within plausible limits [2–11]. In practice, the resort to regularization is just as essential as it is difficult and, in many aspects, it appertains as much to an art as to exact science.

The prevailing approach in acoustics and vibration is surely the popular Tikhonov regularization (control of the energy of the solution) [1,12–17]. A critical aspect of Tikhonov regularization – actually shared by most regularization techniques – is how to automatically determine the amount of regularization to be introduced in the system, which translates into the determination of a "regularization parameter". Several strategies have been developed in this perspective, however, at the present time, there is still no absolutely universal method that is robust enough and always produces good results. Amongst the parameter choice methods used in the field of acoustics and vibration, the Generalized Cross Validation (GCV) [18] and the L-curve [19] seem to prevail largely, although other methods have been investigated such as the Normalized Cumulative Periodogram (NCP) [20,21] and the Morozov discrepancy principle. The latter depends on a good estimate of the measurement noise level, that may not be available in practice. NCP is a relatively recent method whose idea is to track the aspect of the residual error as the regularization parameter changes and select the parameter for which the residual can be considered as close as possible to white noise. Several papers in the literature provide comparisons of different parameter choice methods, either applied in acoustics [10,22–26] or in vibrations [27]. A general conclusion is that the behavior of each method is very problem-dependent and no consensus on which one is the best has been reached. Indeed, the inverse acoustic problem is sometimes so much ill-posed that choosing a proper regularization strategy can make a real difference. Recent publications in acoustics propose different regularization techniques, such as iterative methods (beneficial when dealing with large-scale problems) [28], Tikhonov regularization in its general form (i.e. by the use of discrete smoothing norms) [2], and a sparse regularization techniques [29–31], to cite only a few. Most of them still depend on either a regularization parameter that must be optimally tuned or on a stopping rule for the iterative methods. In a more general

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context (i.e. outside the field of acoustics), Ref. [32] provides an extensive comparison of several parameter choice methods by means of a large simulation study.

This paper introduces a Bayesian approach to regularization that is conceptually rather different from former methodologies that have been prevailing in inverse acoustics. The key idea is to conceive regularization as the introduction of prior information to compensate for the loss of information resulting from the measurement process; this is achieved in the form of a probability density function that characterizes all physically plausible values of the solution before the inverse problem is solved. The solution of the inverse problem – including the reconstruction of the acoustical quantity of interest plus the optimal tuning of the regularization parameter – is then found as the most probable values that comply with both the measurements and the prior information. In the special case of a Gaussian prior – which is investigated only in this paper – the proposed Bayesian regularization scheme boils down to the same structure as Tikhonov regularization, yet with the definite advantage of providing rigorous criteria for automatically tuning the regularization parameter. It is shown in this paper that the proposed Bayesian regularization enjoys several advantages as compared to other criteria traditionally used in inverse acoustics such as GCV and the L-curve:

- for a large range of acoustical configurations (simulation and experiments), it generally returns a regularized solution which is (in the least-square sense) closer to the optimal one,
- for a large range of physical/acoustical parameters (level of noise, frequency range, degree of ill-posedness) it is generally more robust to (strong) additive noise,
- it lends itself to easy implementation for it boils down to searching for the *global minimum* of a 1-D cost function,
- it is fully automatic and does not involve any tuning parameter (it actually *returns* the noise level and expected source energy as byproducts).

These advantages surely deserve a thorough introduction of Bayesian regularization to the acoustical community, even though the mathematical apparatus required may seem far from the acoustical discipline. As a consequence, the first part of the paper (in particular Section 2) is presented as a tutorial before some novel theoretical and experimental results are introduced in other sections. In spite of several precursory Refs. [33–44], the Bayesian regularization does not seem to have attracted much attention in acoustics (possibly because many of the former early works came with complex iterative algorithms). It is part of the present paper aim to partly fill in this gap. The paper also highlights several important properties which, according to the authors' knowledge, have never been recognized before. Part of the present material was first published in Ref. [45], which aimed at finding an optimal basis for source reconstruction and demonstrating the benefit of taking prior information into account within a Bayesian framework. Herein, the focus is on Bayesian regularization only and its generalization to any reconstruction basis, be it optimal or not. The paper contains several original results listed hereafter.

- A theoretical proof is given about the existence of a global minimum of the Bayesian regularization criterion; this property is of prime practical importance since it confers good robustness to regularization (e.g. as compared to GCV and the L-curve for which a global minimum does not exist in general); in addition, it makes possible an automated practice of regularization.
- The posterior probability density function of the regularization parameter is given in the case of complex-valued data (i.e. Fourier transformed data); this is found useful to assess the errors due to regularization in all acoustical quantities of interest and, as far as the authors know, a sensitivity analysis to regularization is demonstrated here for the first time.
- Physical interpretation of Bayesian regularization is given in terms of energy (first principle of thermodynamics) which, hopefully, will participate to bridge the gap between an abstract probabilistic theory and the intuition gained from physics.
- Extensive experimental results are given, both on numerical and on actual data, that clearly demonstrate the supremacy of Bayesian regularization over the GCV and L-curve methods.

The paper is organized as follows. The second section first introduces general facts about the direct acoustic problem and then addresses the inverse problem within the Bayesian probabilistic framework. One objective of this section is to introduce the notations and probabilistic premises necessary for the remaining of the paper, and preferably so in a self-contained treatment of the acoustic inverse problem. As mentioned above, it should be read as a tutorial on the Bayesian approach to inverse (acoustic) problems. The third section addresses the issue of Bayesian regularization, where several new results are established after scrupulously following the Bayesian program. Theoretical developments are accompanied by discussions relating to the properties and practical aspects of the proposed algorithms. The fourth section is an attempt to demonstrate a physical meaning of the proposed Bayesian regularization criteria in terms of thermodynamics. The fifth section addresses the important question as how sensitive the inverse problem is to regularization, to which the Bayesian framework is shown to provide a rather unique answer. Finally, the sixth section is devoted to comparing Bayesian regularization with the state-of-the-art methods, thus demonstrating its asserted superiority.

2. Bayesian approach to the inverse acoustic problem: a tutorial

The object of this section is to cast the inverse acoustic problem within the Bayesian probabilistic framework. This is not only necessary to introduce the fundamental ideas and notations to be used in derivation of the Bayesian regularization criterion in Section 3 (the central result of the paper), but it also offers upstream justification to the classical cost function used in inverse acoustics and its *ad hoc* Tikhonov regularization. Since most of the results presented in this section can be recovered by compiling the Bayesian literature on linear models, it should be read as a tutorial. A general reference on the Bayesian approach to inverse problems is [46]. Moreover, the treatment of the inverse problem could be seen as dual of Bayesian linear regression [47–50] after exchanging the role of the explanatory variables and of the regression coefficients.[1]

2.1. General statement of the inverse acoustic problem

Broadly speaking, the inverse acoustic problem of interest herein amounts to reconstructing a source distribution or "source field" (e.g. parietal pressure or normal velocity) given a finite number of measurements, as typically returned by an array of microphones (or possibly velocity or pressure–velocity probes). More formally, let $q(\mathbf{r}), \mathbf{r} \in \Gamma$, be the source field of interest and Γ its

[1] The objectives of linear regression are, strictly speaking, oriented towards solving the direct and not the inverse problem. However, the inverse problem can be tackled after exchanging the roles of the explanatory variables and of the regression coefficients. Although this might seem artificial at first glance because the regression coefficients are, from a physical point of view, not variables but deterministic "transfer functions", it causes no problem in the Bayesian framework where all parameters are regarded as random.