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Analytical investigations for the design of fast approximation methods for fitting curves and surfaces to scattered data

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Abstract

We present an analytical framework for linear and nonlinear least squares methods and adopt it to the construction of fast iterative methods for fitting curves and surfaces to scattered data. The results are directly applicable to curves and surfaces that have a representation as a linear combination of smooth basis functions associated with the control points. Standard Bézier and B-spline curves / surfaces as well as subdivision schemes have this property. In the global approximation step for the control points our approach couples the standard linear approximation part with the reparameterization to heavily reduce the number of overall steps in the iteration process. This can be formulated in such a way that we have a standard least squares problem in each step. For the local nonlinear parameter corrections our results allow for an optimal choice of the methods used in different stages of the process. Furthermore, regularization terms that express the fairness of the intermediate and / or final result can be added. Adaptivity is easily integrated in our concept. Moreover our approach is well suited for reparameterization occurring in grid generation.

Keywords: Splines; Multivariate approximation; Least squares; Fairing; Numerical analysis

1. Introduction

Fitting curves and surfaces to unorganized point clouds or sample points from given curves/surfaces (aka reparameterization) is an often occurring problem in engineering CAD/CAGD and computer graphics. It has a big history in literature. For a long period surface fitting methods used the distance between a sample point and the corresponding foot point on the fitting surface for minimizing the objective function. We call these methods point distance minimization (PDM). A further, different approach to solve the approximation problem of curves and surfaces is active contour models. The origin of this technique is a paper by Kass et al. [8]. They used a variational formulation of parametric curves for detecting contours in images.

It is well known that the parameterization problem is a fundamental one for the whole approximation process and the final result. Therefore, parameter correction procedures have to be used to improve the quality of the

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final approximation. Unfortunately, the decoupling of the overall fitting procedure into the two independent steps "parameter correction and solving a linear least squares problem with fixed parameters" leads to very slow convergence. To overcome this Pottmann et al. [11] introduced an approach based on the minimization of a quadratic approximant of the squared distance function. An additional aim was to avoid the parameterization problem and to construct algorithms of second order convergence. Methods based on this idea are called SDM (squared distance minimization). Unfortunately, in general the second order Taylor approximant does not lead to symmetric positive definite system matrices and for this reason the existence and uniqueness of the minimum cannot be guaranteed. Thus, the second order Taylor approximation was modified to ensure positive definiteness. However, this modification destroys the second order and thus the claimed quadratic convergence of those methods. Nevertheless such approaches need less iterations. On the other hand the main drawback is a large computational overhead. The curvature computation and the setup of a more complex SDM error function lead to computational inefficiency of SDM. A comparison in [6] shows that the time to attain comparable results used by SDM on iterative optimization is about 30% to 50% more than PDM. Furthermore the parameterization is not really totally avoided.

In [4] the goal of a new development is the avoidance of the curvature computation together with the construction of a cheap error function that accelerates the parameter correction. The algorithm therein can be used for the construction of smooth surfaces from point clouds as well as for the reparameterization of given surfaces. The methods are not restricted to be applied to Bézier or B-spline surfaces. They can be used for subdivision surfaces as well. The focus in that paper is on an analytical understanding of the coupling of parameter correction with the linear least squares approximation step and the transformation of the mathematical model to a form that can be used efficiently for fast solvers. In comparison with the standard PDM approach with decoupling of the linear solver and the parameter correction that method has a tremendous speed up. It needs much less iterations without the drawback of the computational overhead of SDM. Furthermore that method can still be written in the form of a standard least squares problem which is not the case for SDM. Thus, the normal equations and iterative solvers for them can be used as well as orthogonal transformations that have a better condition number than the normal equations. To reduce the number of parameter corrections (the outer loop) a combination of the PDM and the distances of the data points to the linear approximation of the target surface at the projection point is used. The latter one coincides with the squared distance function for points on the surface. The method has superlinear convergence with only a small computational overhead for surface normals. It can be applied to the approximation with standard Bézier- or B-splines as well as with subdivision surfaces.

In this paper we will focus on the necessary parameter correction steps. We will see that different methods have to be used in the different stages consisting of: Initial computation of parameter values, parameter correction during the first steps and parameter correction when the residuals are already small. Furthermore we will see that curvature plays an important role. To get a deeper understanding for the choice of optimal methods for parameter correction steps the possible algorithms are analyzed in detail regarding convergence and error estimation.

The rest of this paper is organized as follows. First we give some basic notations and properties of Bézier, B-spline and subdivision surfaces and the needed basics on approximation with them. Then, we briefly summarize the above-mentioned algorithm from [4]. Next, the main topic is the analytical investigation on the nonlinear approximation algorithms for parameter correction. It will be shown that the theoretically derived behavior is directly reflected in our implementation and leads to an enormous speedup. Finally, a summary and an evaluation regarding computation time and accuracy of the algorithms usually applied for these purposes will be given.

2. Basics on surfaces and approximation

Bézier curves of order n are given by

$$\mathbf{x}(u) = \sum_{i=0}^{n} B_i^n(u)\,\mathbf{p}_i = \sum_{i=0}^{n} \binom{n}{i} u^i (1-u)^{n-i}\,\mathbf{p}_i \tag{1}$$

with $n+1$ control points $\mathbf{p}_i \in \mathbf{R}^d,\ i = 0,\ldots,n$ (here $d = 2,3$). The $B_i^n(u)$ are the Bernstein polynomials. Derivatives of Bézier curves of order n are Bézier curves of order $n-1$. They can be computed from the control points automatically. Another important advantage for interpolation and approximation with curves like (1) is the linearity in the control points $\mathbf{p}_i$. B-spline curves have these properties, too. They are given by

$$\mathbf{x}(u) = \sum_{i=0}^{n} N_i(u)\,\mathbf{p}_i \tag{2}$$