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Domain decomposition based parallel Howard's algorithm

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Abstract

The Howard's algorithm, a technique of resolution for discrete Hamilton-Jacobi equations, is of large use in applications for its high efficiency and good performances. A useful characteristic of the method is the superlinear convergence which, in presence of a finite number of controls, is reached in finite time. Performances of the method can be significantly improved using parallel computing. Building a parallel version of the method is not trivial because of the hyperbolic nature of the problem. In this paper we propose a parallel version of the Howard's algorithm driven by an idea of domain decomposition. This permits to derive some important properties and to prove the convergence under standard assumptions. The good features of the algorithm are shown through some tests and examples.

Keywords: Howard's algorithm (policy iterations); Parallel computing; Domain decomposition

1. Introduction

The *Howard's algorithm* (also known as *policy iteration algorithm*) is a classical method for computing the solution of a *discrete Hamilton–Jacobi (HJ) equation*. This technique, developed by Bellman and Howard [6,19], is of large use in applications thanks to its good properties of efficiency and simplicity.

This algorithm is generally more efficient than other techniques of resolution – the convergence is superlinear and even quadratic in special cases (see [7]) – and always faster than *value iteration* and at least comparable to other iterative methods normally adopted (cf. Section 4, see also [20,23]).

A high efficient alternative is represented by modern *fast techniques* such as Fast Marching [24] and Fast Sweeping [29,30]. Nevertheless those approaches typically require some restrictive hypothesis on the dynamics of the system suffering in consequence of a limited applicability. The policy algorithm instead is extremely general and may be easily adapted to various special cases of interest (cf. Section 5) and to second order problems [11, Section 3.2]. As a drawback, the technique requires the resolution of large linear systems. This can be a difficult step when we approximate equations in spaces of a high dimension. In this paper, we deal with this problem using parallel computing.

The application of parallel computing to HJ equations is a subject of actual interest and recent development. It is in fact an effective tool to overcome difficulties caused by memory storage restrictions and CPU speed limitations in the resolution of real problems.

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In literature, at our knowledge, the first parallel algorithm proposed in the context of HJ equations is [27]. In this paper the authors discuss the numerical solution of the Bellman equation related to an exit time problem for a diffusion process (i.e. for second order elliptic problems). A successive work is [9] in which an operator of semilagrangian kind is described and studied. More recently the issue was discussed in [31] where the authors pass to an equivalent quasi variational inequality and propose a domain decomposition technique. In [8,17] there are presented two different multigrid approaches to obtain a decomposition of the domain in subsets that can be solved independently.

Our approach is slightly different. If we decompose the problem directly in its differential form we can give an easy and consistent interpretation of the condition to impose on the boundaries of the sub-domains. Thereafter we pass to a discrete version of such decomposed problem. Now it is easy to show the convergence of the algorithm to the discrete solution.

The paper is structured as follows: in Section 2 we recall the Howard's algorithm and the relation with the differential problem focusing on the case of its optimal control interpretation. In Section 3 we present the algorithm and we study the convergence. Section 4 is dedicated to test the performances and to show the advantages and speed-up factors with respect to the non parallel version. We end by presenting some possible extensions of the technique to some problems of interest: target problems, obstacle avoidance and max–min problems.

2. Howard's algorithm

The problem considered is the following. Let be Ω bounded open domain of $\mathbb{R}^d$ ($d \geq 1$); the steady, first order HJ equation is:

$$\begin{cases} \lambda v(x) + H(x, Dv(x)) = 0 & x \in \Omega, \\ v(x) = g(x) & x \in \partial\Omega, \end{cases} \tag{1}$$

where, following the *optimal control interpretation*, $\lambda \in \mathbb{R}^+$ is the *discount factor*, $g : \Omega \to \mathbb{R}$ is the *exit cost*, and the *Hamiltonian* $H : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is defined by: $H(x, p) := \inf_{\alpha\in\mathcal{A}}\{-f(x, \alpha) \cdot p - l(x, \alpha)\}$ with $f : \Omega \times \mathcal{A} \to \mathbb{R}$ (*dynamics*), $l : \Omega \times \mathcal{A} \to \mathbb{R}$ (*running cost*) and $\mathcal{A}$ a compact set. The choice of such Hamiltonian is not restrictive but useful to simplify the presentation. In Section 5 we extend the study to other Hamiltonians.

Under classical assumptions on the data (we can suppose $f(\cdot, \cdot)$ and $l(\cdot, \cdot)$ continuous, $f(\cdot, \alpha)$ and $l(\cdot, \alpha)$ Lipschitz continuous for all $\alpha \in \mathcal{A}$ and the *Soner's condition* [25] is verified), it is known (see i.e. [2, Th.3.1]) that Eq. (1) admits a unique continuous solution $v : \overline{\Omega} \to \mathbb{R}$ in the *viscosity solutions* sense.

The solution v is the value function of the infinite horizon problem with exit cost, where τ_x is the *first time of exit* from Ω:

$$v(x) = \inf_{a(\cdot)\in L^\infty([0,+\infty[;\,\mathcal{A})} \int_0^{\tau_x(a)} l(y_x(s), a(s))e^{-\lambda s}\, ds + e^{-\lambda\tau_x(a)} g(y_x(\tau_x(a))),$$

$$\text{where } y_x(\cdot) \text{ is a.e. solution of } \begin{cases} \dot{y}(t) = f(y(t), a(t)) \\ y(0) = x. \end{cases}$$

Many numerical schemes for the approximation of this problem have been proposed. Let us mention Finite Differences Schemes [13,26], semilagrangian [14], Discontinuous Galerkin [12] and many others. In this paper we focus on a *monotone*, *consistent* and *stable* scheme (wide class including the first two mentioned above) for the approximation of (1).

Considered a discrete grid G with N points x_j, $j = 1, \ldots, N$ on the domain $\overline{\Omega}$, the finite N-dimensional approximation of v, V is the solution of the discrete equation ($V_j = V(x_j)$)

$$F_i^h(V_1, \ldots, V_N) = F_i^h(V) = 0, \quad i \in \{1, \ldots, N\}, \tag{2}$$

where $h := \max diam S_j$ (biggest diameter of the family of simplices S_j on G) is the discretization parameter. The Dirichlet conditions of (1) are

$$F_j^h(V_1, \ldots, V_N) := V_j - g(x_j), \quad x_j \in \partial\Omega. \tag{3}$$

We assume on F some standard hypotheses:

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