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Original Articles

High-order filtered schemes for first order time dependent linear and non-linear partial differential equations

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Abstract

This work is based on high-order "filtered scheme". Recently filtered scheme has been introduced to solve some first order Hamilton–Jacobi equations. In this paper, we aim to solve some linear and non-linear partial differential equations by a high order filtered scheme. The proposed filtered scheme that is not monotone but still satisfies some ϵ-monotone property with a convergence result and with precise error estimate also has been proven. We will present filtration of different schemes for some linear and non-linear partial differential equations in several dimensions.

Keywords: Hamilton–Jacobi equation; High-order schemes; ϵ-monotone scheme; Semi-Lagrangian schemes; Viscosity solutions

1. Introduction

In this paper, we aim to solve first order time dependent partial differential equations (PDEs) in particular hyperbolic conservation law and Hamilton–Jacobi (HJ) equation by high-order filtered scheme. It is well known that, in 1D, there is a strong link between time-dependent HJ equations and hyperbolic conservation laws. To be more precise, the viscosity solution of the evolutive HJ equation is the primitive of entropy solution of the corresponding conservation law. Due to this link several schemes have been developed to solve hyperbolic conservation law (see Refs. [8,12–14]) and many of them extended for HJ equations. For instance, well-known high-order essentially non-oscillatory (ENO) scheme has been introduced by A. Harten et al. in [15] for conservation laws, and then extended to HJ equation by Osher and Shu [17]. ENO schemes have been shown to have high-order accuracy numerically however no general convergence results are available. The interest for these schemes is due to the fact that they should be high-order accurate if they converge. In [2], Barles and Souganidis have given a general frame work for the convergence of approximated solution towards the viscosity solution under generic monotonicity, stability and consistency assumptions. Recently filter scheme has been introduced in [11] to solve Monge–Ampere equation, and

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adapted for the stationary and time-dependent first order HJ equations in [3,4,16,18]. Proposed scheme in [4] is written in explicit time marching form ("fully explicit" schemes) which is well adapted to time-dependent equations, while the setting of [11] or [16] can be better adapted to solve stationary equations. In our work, we follow the filtered scheme from [4]. This framework enables the development of simple schemes that have high-order consistency in both space and time. Filter can stabilize an unstable scheme and achieves higher-order accuracy. It is well known by the Godunov theorem that monotone scheme can be at most first order hence one has to look for the non-monotonicity. Then it is difficult to combine non-monotonicity and converges to the viscosity solution. In [4], convergence results and the error estimate have been proved for stationary and time-dependent HJ equations.

In this paper, we present several examples with filtration of different schemes up to 3D. For the monotone scheme we will use semi-Lagrangian (SL) schemes (by Courant, Isaacson and Rees [6]) and finite difference scheme (by Crandall and Lions in [8] with the convergence result) for HJ equations. For high-order scheme we will use second and third order schemes. We will compare the proposed filtered scheme with the high-order scheme used in filtration and ENO scheme via several numerical tests up to 3D.

Organization of paper. In Section 2, we will present the model problem and recall filtered scheme from [4] with the limiter. In Section 3, we will present some numerical examples of second and third order filtered scheme up to three-dimensions. In Section 4, we will conclude and finally Appendix contains some theoretical outline.

2. Filtered scheme

We recall the filtered scheme from [4] for the following model problem:

$$\partial_t v + H(x, \nabla v) = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^d \tag{1}$$

$$v(0, x) = v_0(x), \quad x \in \mathbb{R}^d, \tag{2}$$

the typical assumptions on Hamiltonian H and the initial data $v_0(x)$ are:

A1. $H(\cdot, \cdot, \cdot)$ is uniformly continuous in all the variables.
A2. $H(x, v, \cdot)$ is convex and coercive.
A3. $H(x, \cdot, \nabla v)$ is monotone.
A4. $v_0(x)$ is Lipschitz continuous.

The above assumptions guarantee existence and uniqueness in the framework of weak solutions in viscosity sense [1,7]. For simplicity, we present scheme in 1D and can be easily adapted to the higher dimension (filtered scheme for 2D has been presented in [18]). The basic idea of filter scheme is the combination of the low order and high-order schemes. This allows us to construct finite difference schemes which are easy to implement and behave like a monotone scheme in the singular region and as a high-order scheme where the solution is smooth. We use the discontinuous filter function which has been used in [4,16,18] for which the filtered scheme is still an "ϵ-monotone" scheme (see (17)). In our case, we justify the use of this discontinuous filter to obtain a high order numerical behavior of the scheme in the L^∞ norm. We observe that using instead the continuous filter initially introduced in [11] leads to only first order behavior although for steady equations both filters give similar results.

Discretization: Let $\Delta t > 0$ be a time step (in the form of $\Delta t = \frac{T}{N}$ for some $N \geq 1$), and $\Delta x > 0$ be a space step. A uniform mesh in time is defined by $t_n := n\Delta t$, $n \in [0, \ldots, N]$, and in space by the nodes $x_j := j\Delta x$, $j \in \mathbb{Z}$. Hence the filtered scheme (for more details see [4]) is then defined as

$$u_j^{n+1} \equiv S^F(u^n)_j := S^M(u^n)_j + \epsilon \Delta t F\left(\frac{S^A(u^n)_j - S^M(u^n)_j}{\epsilon \Delta t}\right), \tag{3}$$

where $\epsilon = \epsilon_{\Delta t, \Delta x} > 0$ is a parameter satisfying

$$\lim_{(\Delta t, \Delta x) \to 0} \epsilon = 0 \tag{4}$$

where S^M is a monotone scheme here we will consider two cases for the monotone schemes.

• **Case 1**: S^M is based on a first order finite difference scheme [8]. Hence the monotone finite difference scheme written as

$$S^M(u^n)_j := S^M(u^n)(x_j) := u_j^n - \Delta t\, h^M(x_j, D^- u_j^n, D^+ u_j^n), \qquad D^\pm u_j^n := \pm \frac{u_{j\pm1}^n - u_j^n}{\Delta x}, \tag{5}$$

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