Available online at www.sciencedirect.com
ScienceDirect
Mathematics and Computers in Simulation ▮ (▮▮▮▮) ▮▮▮–▮▮▮
MATHEMATICS AND COMPUTERS IN SIMULATION
www.elsevier.com/locate/matcom

Original Articles

A POD-based reduced-order Crank–Nicolson finite volume element extrapolating algorithm for 2D Sobolev equations[☆]

Zhendong Luo[a,*], Fei Teng[b], Jing Chen[c,*]

[a] School of Mathematics and Physics, North China Electric Power University, Beijing 102206, China
[b] School of Control and Computer Engineering, North China Electric Power University, Beijing 102206, PR China
[c] College of Science, China Agricultural University, Beijing 100083, China

Received 20 November 2013; received in revised form 6 July 2017; accepted 7 November 2017
Available online xxxxx

Abstract

Based on proper orthogonal decomposition (POD), a new type of reduced-order Crank–Nicolson finite volume element extrapolating algorithm (CNFVEEA) including very few degrees of freedom but holding fully second-order accuracy for two-dimensional (2D) Sobolev equations is established firstly. Then, the error estimates of POD-based reduced-order CNFVEEA solutions are provided, which acted as a suggestion for choosing number of POD basis and a criterion for updating POD basis, and the procedure for the implementation of the POD-based reduced-order CNFVEEA is given. Finally, a numerical example is presented illustrating that the numerical computational conclusions are consistent with theoretical ones. Moreover, it is shown that the POD-based reduced-order CNFVEEA is very suitable to finding numerical solutions of 2D Sobolev equations and is better than the POD-based FVE formulation with first-order accuracy in time.

Keywords: Proper orthogonal decomposition; Reduced-order Crank–Nicolson finite volume element extrapolating algorithm; Two-dimensional Sobolev equations; Error estimate

1. Introduction

Let $\Omega \subset R^2$ be a bounded polygonal domain with boundary $\partial\Omega$. The two-dimensional (2D) Sobolev equations are described as the following initial boundary value problem.

Problem I. Find u such that

$$u_t - \varepsilon \Delta u_t - \gamma \Delta u = f, \; (x, y, t) \in \Omega \times (0, T], \tag{1.1}$$

☆ This work is jointly supported by the National Science Foundation of China (11671106) and the Fundamental Research Funds for the Central Universities (2017XS067).

* Corresponding authors.
E-mail addresses: zhdluo@163.com (Z.D. Luo), jing_quchen@163.com (J. Chen).

https://doi.org/10.1016/j.matcom.2017.11.002
0378-4754/

$$u(x, y, t) = \psi(x, y, t),\ (x, y, t) \in \partial\Omega \times (0, T], \tag{1.2}$$

$$u(x, y, 0) = \varphi_0(x, y),\ (x, y) \in \Omega, \tag{1.3}$$

where $u_t = \partial u/\partial t$, ε and γ are two given positive coefficients, $f(x, y, t)$ is the source term, $\psi(x, y, t)$ the boundary value function, $\varphi_0(x, y)$ the initial function, and T the time upper bound. As a matter of convenience and without loss of generality, the boundary value function $\psi(x, y, t)$ may be assumed a zero function in the following theoretical study.

Sobolev equations (1.1)–(1.3) constitute a significant system of equations with real physical background. They can be used to describe the fluid flow penetrating rocks, soils, or different viscous media (see [4,50,55]). Due to the system of equations including complicated computing domains, initial value functions, or source functions in real-world engineering problems, there is no analytical solution in general. One has to rely on numerical solutions (see, e.g., [8,50,55]). The finite volume element (FVE) method (see [7,52]) is considered as one of the most effective discretization approaches for the 2D Sobolev equations. Compared to the finite element (FE) and finite difference methods, the FVE method is generally easier to implement and offer flexibility in handling complicated computing domains. Especially, the FVE method can ensure local mass conservation and a highly desirable property in many applications. Therefore, it is as the preferred numerical approach. It is also referred to as a box method (see [3,18]) or a generalized difference method (see [23,24]). It has been widely used in sciences and engineering computations (see, e.g., [3,5,7,12,13,18,22–24,34,49,52,56,57]).

Though, based on the trial function space formed by piecewise linear polynomial, some FVE methods (see [8,25]) have been presented for the 2D Sobolev equations, they only hold first-order convergence. In order to ameliorate the convergence in literatures [8,25], by using the same trial function space as in [25] and adopting a Crank–Nicolson (CN) technique to discrete time, a CN FVE (CNFVE) formulation with fully second-order accuracy about time and spacial variables is established for the 2D Sobolev equations (see [29]). Even if the CNFVE formulation in [29] is one order higher than those in [8,25], it includes many degrees of freedom, too. Thus, a critical issue is how greatly to reduce its degrees of freedom so as to alleviate calculating load and save calculation time and memory requirements in the process of calculation but keep sufficiently high accuracy of numerical solutions.

The proper orthogonal decomposition (POD) method (see [15–17]) is an effective means that can reduce degrees of freedom (unknown quantities) of numerical models for time-dependent partial differential equations so as to alleviate calculating load and the accumulation of truncation errors in the computational process. It is mainly to find an orthonormal basis for the known data under the least squares sense, i.e., to find optimal order approximations for the known data.

Though some POD-based reduced-order numerical models have been established (see, e.g., [2,11,9,10,20,21,19, 26,27,30–33,35–38,40–42,39,43–46,48,51,53]), to the best of our knowledge, there is not any report that the CNFVE method for the 2D Sobolev equations is reduced-order by means of the POD method. Especially, the most existing POD-based reduced-order models as mentioned above were established via the POD basis formulated by the classical numerical solutions over the total time span $[0, T]$, before recomputing the solutions over the same time span $[0, T]$, which are unrewarding repeated computations. Therefore, we here improve the existing POD-based reduced-order models, i.e., only use the first few numerical solutions over the very short time span $[0, T_0]$ ($T_0 \ll T$) to formulate the POD basis and establish a POD-based reduced-order CNFVE extrapolation algorithm (CNFVEEA) including very few degrees of freedom but holding fully second-order accuracy, before finding the numerical solutions over the total time span $[0, T]$ for the 2D Sobolev equations via extrapolation and iteration, as well as POD basis updating. Thus, we sufficiently exploit the merit of POD technique, i.e., utilize the known data (on the very short time span $[0, T_0]$ and $T_0 \ll T$) to forecast (or infer) future physic phenomenons (on the time span $[T_0, T]$), which is why we name this reduced-order method as the reduced-order CNFVEEA. We also deduce the error estimates between the POD-based reduced-order CNFVEEA solutions and the classical CNFVE solutions, which acted as a suggestion for choosing number of POD basis and a criterion for updating POD basis, and give a procedure for the implementation of the POD-based reduced-order CNFVEEA. We use a numerical example to verify that the numerical computational conclusions are consistent with theoretical ones, too. This signifies that the POD-based reduced-order CNFVEEA is suitable for solving the 2D Sobolev equations.

Though the POD-based reduced-order FVE method (see [39]) and CNFVE formulation (see [35]) for the 2D parabolic equations, the POD-based reduced-order semi-discrete FVE algorithm (see [37]) and fully-discrete FVE method (see [26]) for the 2D viscoelastic equations, and the POD-based reduced-order FVE method for the 2D solute